Constrained energy problems with external fields for infinite dimensional vector measures

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Abstract. We consider a constrained minimal energy problem with an external field over noncompact classes of infinite dimensional vector measures $(\mu^i)_{i\in I}$ on a locally compact space. The components μ^i are positive measures with the properties $\int g_i\,d\mu^i = a_i$ and $\sigma^i - \mu^i \geqslant 0$ (where a_i, g_i, and σ^i are given) and supported by closed sets A_i with the sign $+1$ or -1 prescribed such that $A_i \cap A_j = \varnothing$ whenever $\operatorname{sign} A_i \neq \operatorname{sign} A_j$, and the law of interaction of μ^i, $i \in I$, is determined by the matrix $\bigl(\operatorname{sign} A_i \operatorname{sign} A_j\bigr)_{i,j\in I}$. For all positive definite kernels satisfying Fuglede's condition of consistency between the vague ($=$weak*) and strong topologies, sufficient conditions for the existence of minimizers are established and their uniqueness and vague compactness are studied. Examples illustrating the sharpness of the sufficient conditions are provided. We also analyze continuity properties of minimizers in the vague and strong topologies when A_i and σ^i are varied simultaneously. The results are new even for classical kernels in $\mathbb{R}^n$, which is important in applications.

1. Introduction

In all that follows, X denotes a locally compact Hausdorff space and $\mathfrak{M} = \mathfrak{M}(\mathrm{X})$ the linear space of all real-valued scalar Radon measures ν on X equipped with the *vague* ($=$ *weak**) topology, i.e., the topology of pointwise convergence on the class $\mathrm{C}_0(\mathrm{X})$ of all real-valued continuous functions φ on X with compact support.

A *kernel* κ on X is meant to be an element from $\Phi(\mathrm{X}\times\mathrm{X})$, where $\Phi(\mathrm{Y})$ consists of all lower semicontinuous functions $\psi : \mathrm{Y} \to (-\infty, \infty]$ such that $\psi \geqslant 0$ unless Y is compact. Given $\nu, \nu_1 \in \mathfrak{M}$, the *mutual energy* and the *potential* relative to the kernel κ are defined by

$$\kappa(\nu,\nu_1) := \int \kappa(x,y)\,d(\nu\otimes\nu_1)(x,y) \quad \text{and} \quad \kappa(\,\cdot\,,\nu) := \int \kappa(\,\cdot\,,y)\,d\nu(y),$$

respectively. (When introducing notation, we always tacitly assume the corresponding object on the right to be well defined — as a finite number or $\pm\infty$.)

For $\nu = \nu_1$ the mutual energy $\kappa(\nu, \nu_1)$ defines the *energy* $\kappa(\nu, \nu)$ of ν. We denote by $\mathcal{E} = \mathcal{E}_\kappa(\mathrm{X})$ the set of all $\nu \in \mathfrak{M}$ with $-\infty < \kappa(\nu, \nu) < \infty$.

We shall mainly be concerned with a *positive definite* kernel κ, which means that it is symmetric (i.e., $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in \mathrm{X}$) and the energy $\kappa(\nu, \nu)$, $\nu \in \mathfrak{M}$, is nonnegative whenever defined. Then $\mathcal{E}$ forms a pre-Hilbert space with the scalar product $\kappa(\nu, \nu_1)$ and the seminorm $\|\nu\|_{\mathcal{E}} := \|\nu\|_\kappa := \sqrt{\kappa(\nu, \nu)}$ (see [12]); the topology on $\mathcal{E}$, determined by this seminorm, is called *strong*. A positive definite kernel κ is *strictly positive definite* if the seminorm $\|\cdot\|_{\mathcal{E}}$ is a norm.

Given a closed set $F \subset \mathrm{X}$, we denote by $\mathfrak{M}^+(F)$ the convex cone of all nonnegative $\nu \in \mathfrak{M}$ supported by F, and let $\mathcal{E}^+(F) := \mathfrak{M}^+(F) \cap \mathcal{E}$. Also write $\mathfrak{M}^+ := \mathfrak{M}^+(\mathrm{X})$ and $\mathcal{E}^+ := \mathcal{E}^+(\mathrm{X})$.

We consider a countable, locally finite collection $\mathbf{A} = (A_i)_{i \in I}$ of fixed closed sets $A_i \subset \mathrm{X}$ with the sign $+1$ or -1 prescribed such that the oppositely signed sets are mutually disjoint. Let $\mathfrak{M}^+(\mathbf{A})$ stand for the Cartesian product $\prod_{i \in I} \mathfrak{M}^+(A_i)$; then an element $\boldsymbol{\mu}$ of $\mathfrak{M}^+(\mathbf{A})$ is a (nonnegative) *vector measure* $(\mu^i)_{i \in I}$ with the components $\mu^i \in \mathfrak{M}^+(A_i)$. The topology of the product space $\prod_{i \in I} \mathfrak{M}^+(A_i)$, where every $\mathfrak{M}^+(A_i)$ is endowed with the vague topology, is likewise called *vague*.

If a vector measure $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ and a vector-valued function $\mathbf{u} = (u_i)_{i \in I}$ with μ^i-measurable components $u_i : A_i \to [-\infty, \infty]$ are given, then for brevity we write[1]

$$\langle \mathbf{u}, \boldsymbol{\mu} \rangle := \sum_{i \in I} \int u_i \, d\mu^i.$$

Let a kernel κ be fixed. In accordance with an electrostatic interpretation of a condenser, we assume that the interaction between the charges lying on the conductors A_i, $i \in I$, is characterized by the matrix $(\alpha_i \alpha_j)_{i,j \in I}$, where $\alpha_i := \operatorname{sign} A_i$. Then the *energy* of $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ is defined by the formula

$$\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) := \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu^j).$$

We denote by $\mathcal{E}^+(\mathbf{A})$ the set of all $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ with $-\infty < \kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) < \infty$.

Also fix a vector-valued function $\mathbf{f} = (f_i)_{i \in I}$, treated as an *external field*, and assume it to satisfy one of the following two cases:

Case I. $f_i \in \Phi(\mathrm{X})$ *for all* $i \in I$;

Case II. $f_i = \alpha_i \kappa(\cdot, \zeta)$ *for all* $i \in I$, *where* $\zeta \in \mathcal{E}$ *is a signed measure*.

Furthermore, suppose each f_i to affect the charges on A_i only; then the $\mathbf{f}$-*weighted energy* of $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ is given by the expression

$$G_{\mathbf{f}}(\boldsymbol{\mu}) := \kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) + 2\langle \mathbf{f}, \boldsymbol{\mu} \rangle. \tag{1.1}$$

Let $\mathcal{E}^+_{\mathbf{f}}(\mathbf{A})$ consist of all $\boldsymbol{\mu} \in \mathcal{E}^+(\mathbf{A})$ with $-\infty < G_{\mathbf{f}}(\boldsymbol{\mu}) < \infty$.

[1] Here and in the sequel, an expression $\sum_{i \in I} c_i$ is meant to be well defined provided so is every summand c_i and the sum does not depend on the order of summation — though might be $\pm\infty$. Then, by Riemann series theorem, the sum is finite if and only if the series converges absolutely.

Also fix a vector measure $\boldsymbol{\sigma} \in \mathfrak{M}^+(\mathbf{A})$, serving as a *constraint*, a numerical vector $\mathbf{a} = (a_i)_{i\in I}$ with $a_i > 0$ for all $i \in I$, and a vector-valued function $\mathbf{g} = (g_i)_{i\in I}$, where all the $g_i : A_i \to (0,\infty)$ are continuous. In the study, we are interested in the problem of minimizing $G_{\mathbf{f}}(\boldsymbol{\mu})$ over the class of all $\boldsymbol{\mu} \in \mathcal{E}^+_{\mathbf{f}}(\mathbf{A})$ with the properties that $\langle g_i, \mu^i\rangle = a_i$ and $\sigma^i - \mu^i \geqslant 0$ for all $i \in I$.

Along with its electrostatic interpretation, such a problem has found various important applications in approximation theory (see, e.g., [8, 9, 21]).

The main question is whether minimizers $\lambda^{\boldsymbol{\sigma}}_{\mathbf{A}}$ in the constrained minimal $\mathbf{f}$-weighted energy problem exist. If $\mathbf{A}$ is finite, all the A_i are compact, $\kappa(x, y)$ is continuous on $A_\ell \times A_j$ whenever $\alpha_\ell \neq \alpha_j$, and Case I takes place, then the existence of those $\lambda^{\boldsymbol{\sigma}}_{\mathbf{A}}$ can easily be established by exploiting the vague topology only, since then the class of admissible vector measures is vaguely compact, while $G_{\mathbf{f}}(\boldsymbol{\mu})$ is vaguely lower semicontinuous (cf. [14, 19, 20, 22]).

However, these arguments break down if any of the above-mentioned four assumptions is dropped, and then the problem on the existence of minimizers becomes rather nontrivial. In particular, the class of admissible vector measures is no longer vaguely compact if any of the A_i is noncompact. Another difficulty is that $G_{\mathbf{f}}(\boldsymbol{\mu})$ might not be vaguely lower semicontinuous when Case II holds.

To solve the problem on the existence of minimizers $\lambda^{\boldsymbol{\sigma}}_{\mathbf{A}}$ in the general case, we restrict ourselves to a positive definite kernel κ and develop an approach based on the following crucial arguments.

The set $\mathcal{E}^+(\mathbf{A})$ is shown to be a *semimetric* space with the semimetric

$$\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{E}^+(\mathbf{A})} := \Big[\sum_{i,j\in I} \alpha_i\alpha_j\kappa(\mu^i_1 - \mu^i_2, \mu^j_1 - \mu^j_2)\Big]^{1/2}, \tag{1.2}$$

and one can define an inclusion R of $\mathcal{E}^+(\mathbf{A})$ into the pre-Hilbert space $\mathcal{E}$ such that $\mathcal{E}^+(\mathbf{A})$ becomes *isometric* to its R-image, the latter being regarded as a semimetric subspace of $\mathcal{E}$ (see Theorem 3.11). Similar to the terminology in $\mathcal{E}$, we therefore call the topology of the semimetric space $\mathcal{E}^+(\mathbf{A})$ *strong*.

Another crucial fact is that, for rather general κ, $\mathbf{g}$, and $\mathbf{a}$, the topological subspace of $\mathcal{E}^+(\mathbf{A})$ consisting of all $\boldsymbol{\mu}$ such that $\langle g_i, \mu^i\rangle \leqslant a_i$ and $\sigma^i - \mu^i \geqslant 0$ for all $i \in I$ turns out to be *strongly complete* (see Theorem 7.4).

Using these arguments, we obtain sufficient conditions for the existence of minimizers $\lambda^{\boldsymbol{\sigma}}_{\mathbf{A}}$ and establish statements on their uniqueness and vague compactness (see Lemma 4.1 and Theorem 6.2). Examples illustrating the sharpness of the sufficient conditions are provided (see Sec. 12). We also analyze continuity properties of $\lambda^{\boldsymbol{\sigma}}_{\mathbf{A}}$ relative to the vague and strong topologies when both $\boldsymbol{\sigma}$ and $\mathbf{A}$ are varied (see Theorems 6.7, 6.9 and Corollaries 6.8, 6.10).

The results obtained hold true, e.g., for the Newtonian, Green or Riesz kernels in $\mathbb{R}^n$, $n \geqslant 2$, as well as for the restriction of the logarithmic kernel in $\mathbb{R}^2$ to the open unit disk, which is important in applications.

2. Preliminaries: topologies, consistent and perfect kernels

In all that follows, we suppose the kernel κ to be positive definite. In addition to the *strong* topology on $\mathcal{E}$, determined by the seminorm $\|\nu\| := \|\nu\|_{\mathcal{E}} := \|\nu\|_\kappa := \sqrt{\kappa(\nu,\nu)}$, it is often useful to consider the *weak* topology on $\mathcal{E}$, defined by means of the seminorms $\nu \mapsto |\kappa(\nu,\mu)|$, $\mu \in \mathcal{E}$ (see [12]). The Cauchy–Schwarz inequality

$$|\kappa(\nu,\mu)| \leqslant \|\nu\| \, \|\mu\|, \quad \text{where } \nu,\mu \in \mathcal{E},$$

implies immediately that the strong topology on $\mathcal{E}$ is finer than the weak one.

In [12, 13], B. Fuglede introduced the following two *equivalent* properties of consistency between the induced strong, weak, and vague topologies on $\mathcal{E}^+$:

(C_1) *Every strong Cauchy net in $\mathcal{E}^+$ converges strongly to any of its vague cluster points;*

(C_2) *Every strongly bounded and vaguely convergent net in $\mathcal{E}^+$ converges weakly to the vague limit.*

Definition 2.1. Following Fuglede [12], we call a kernel κ *consistent* if it satisfies either of the properties (C_1) and (C_2), and *perfect* if, in addition, it is strictly positive definite.

Remark 2.2. One has to consider *nets* or *filters* in $\mathfrak{M}^+$ instead of sequences, since the vague topology in general does not satisfy the first axiom of countability. We follow Moore's and Smith's theory of convergence, based on the concept of nets (see [18]; cf. also [11, Chap. 0] and [16, Chap. 2]). However, if X is metrizable and countable at infinity, then $\mathfrak{M}^+$ satisfies the first axiom of countability (see [12, Lemma 1.2.1]) and the use of nets may be avoided.

Theorem 2.3 (Fuglede [12]). *A kernel κ is perfect if and only if $\mathcal{E}^+$ is strongly complete and the strong topology on $\mathcal{E}^+$ is finer than the vague one.*

Remark 2.4. In $\mathbb{R}^n$, $n \geqslant 3$, the Newtonian kernel $|x-y|^{2-n}$ is perfect [5]. So are the Riesz kernels $|x-y|^{\alpha-n}$, $0<\alpha<n$, in $\mathbb{R}^n$, $n \geqslant 2$ [6, 7], and the restriction of the logarithmic kernel $-\log|x-y|$ in $\mathbb{R}^2$ to the open unit disk [17]. Furthermore, if D is an open set in $\mathbb{R}^n$, $n \geqslant 2$, and its generalized Green function g_D exists (see, e.g., [15, Th. 5.24]), then the kernel g_D is perfect as well [10].

Remark 2.5. As is seen from the above definitions and Theorem 2.3, the concept of consistent or perfect kernels is an efficient tool in minimal energy problems over classes of *nonnegative scalar* Radon measures with finite energy. Indeed, the theory of capacities of *sets* has been developed in [12] for exactly those kernels. We shall show below that this concept is efficient, as well, in minimal energy problems over classes of *vector measures* of finite or infinite dimensions. This is guaranteed by a theorem on the strong completeness of proper subspaces of the semimetric space $\mathcal{E}^+(\mathbf{A})$, to be stated in Sec. 7.2.

3. Condensers. Vector measures and their energies

3.1. Condensers of countably many plates. Associated vector measures

Throughout the article, let I^+ and I^- be fixed countable, disjoint sets of indices, where the latter is allowed to be empty, and let I denote their union. Assume that to every $i \in I$ there corresponds a (unique) nonempty, closed set $A_i \subset \mathrm{X}$.

Definition 3.1. A collection $\mathbf{A} = (A_i)_{i\in I}$ is called an (I^+, I^-)-*condenser* (or simply a *condenser*) in X if every compact subset of X intersects with at most finitely many A_i and

$$A_i \cap A_j = \varnothing \quad \text{for all} \;\; i \in I^+,\; j \in I^-. \tag{3.1}$$

A condenser $\mathbf{A}$ is called *compact* if so are all A_i, $i \in I$, and *finite* if I is finite. The sets A_i, $i \in I^+$, and A_j, $j \in I^-$, are called the *positive* and, respectively, the *negative plates* of $\mathbf{A}$. (Note that any two equally signed plates can intersect each other or even coincide.) In the sequel, also the following notation will be used:

$$A^+ := \bigcup_{i\in I^+} A_i, \qquad A^- := \bigcup_{i\in I^-} A_i.$$

Observe that A^+ and A^- might both be noncompact even for a compact $\mathbf{A}$.

Given a condenser $\mathbf{A}$, let $\mathfrak{M}^+(\mathbf{A})$ consist of all nonnegative vector measures $\boldsymbol{\mu} = (\mu^i)_{i\in I}$, where $\mu^i \in \mathfrak{M}^+(A_i)$ for all $i \in I$; that is, $\mathfrak{M}^+(\mathbf{A}) := \prod_{i\in I} \mathfrak{M}^+(A_i)$. The product topology on $\mathfrak{M}^+(\mathbf{A})$, where every $\mathfrak{M}^+(A_i)$ is equipped with the vague topology, is likewise called *vague*. Since the space $\mathfrak{M}(\mathrm{X})$ is Hausdorff, so is $\mathfrak{M}^+(\mathbf{A})$ (cf. [16, Chap. 3, Th. 5]).

A set $\mathfrak{F} \subset \mathfrak{M}^+(\mathbf{A})$ is *vaguely bounded* if, for every $\varphi \in \mathrm{C}_0(\mathrm{X})$ and every $i \in I$,

$$\sup_{\mu\in\mathfrak{F}} |\mu^i(\varphi)| < \infty.$$

Lemma 3.2. *If $\mathfrak{F} \subset \mathfrak{M}^+(\mathbf{A})$ is vaguely bounded, then it is vaguely relatively compact.*

Proof. Since by [2, Chap. III, § 2, Prop. 9] any vaguely bounded part of $\mathfrak{M}$ is vaguely relatively compact, the lemma follows from Tychonoff's theorem on the product of compact spaces (see, e.g., [16, Chap. 5, Th. 13]). □

3.2. Mapping $R : \mathfrak{M}^+(\mathbf{A}) \to \mathfrak{M}$. Relation of R-equivalency on $\mathfrak{M}^+(\mathbf{A})$

Since each compact subset of X intersects with at most finitely many A_i, for every $\varphi \in \mathrm{C}_0(\mathrm{X})$ only a finite number of $\mu^i(\varphi)$ (where $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ is given) are nonzero. This yields that to every vector measure $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ there corresponds a unique scalar Radon measure $R\boldsymbol{\mu} \in \mathfrak{M}$ such that

$$R\boldsymbol{\mu}(\varphi) = \sum_{i\in I} \alpha_i \mu^i(\varphi) \quad \text{for all} \;\; \varphi \in \mathrm{C}_0(\mathrm{X}),$$

where

$$\alpha_i := \begin{cases} +1 & \text{if} \quad i \in I^+, \\ -1 & \text{if} \quad i \in I^-. \end{cases}$$

Then, because of (3.1), the positive and negative parts in the Hahn–Jordan decomposition of $R\boldsymbol{\mu}$ can respectively be written in the form

$$R\boldsymbol{\mu}^+ = \sum_{i\in I^+} \mu^i, \qquad R\boldsymbol{\mu}^- = \sum_{i\in I^-} \mu^i.$$

Of course, the inclusion R of $\mathfrak{M}^+(\mathbf{A})$ into $\mathfrak{M}$, thus defined, is in general non-injective, i.e., one may choose $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathfrak{M}^+(\mathbf{A})$ so that $\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$, though $R\boldsymbol{\mu}_1 = R\boldsymbol{\mu}_2$. We call $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathfrak{M}^+(\mathbf{A})$ *R-equivalent* if $R\boldsymbol{\mu}_1 = R\boldsymbol{\mu}_2$ — or, which is equivalent, whenever $\sum_{i\in I} \mu_1^i = \sum_{i\in I} \mu_2^i$.

Observe that the relation of R-equivalency implies that of identity (and, hence, these two relations on $\mathfrak{M}^+(\mathbf{A})$ are actually equivalent) if and only if all A_i, $i \in I$, are mutually disjoint.

Lemma 3.3. *The vague convergence of $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathfrak{M}^+(\mathbf{A})$ to $\boldsymbol{\mu}_0 \in \mathfrak{M}^+(\mathbf{A})$ implies the vague convergence of $(R\boldsymbol{\mu}_s)_{s\in S}$ to $R\boldsymbol{\mu}_0$.*

Proof. This is obvious in view of the fact that the support of any $\varphi \in \mathrm{C}_0(\mathrm{X})$ can have points in common with only finitely many A_i. □

Remark 3.4. Lemma 3.3 in general can not be inverted. However, if all the A_i are mutually disjoint, then the vague convergence of $(R\boldsymbol{\mu}_s)_{s\in S}$ to $R\boldsymbol{\mu}_0$ implies the vague convergence of $(\boldsymbol{\mu}_s)_{s\in S}$ to $\boldsymbol{\mu}_0$, which is seen by using the Tietze–Urysohn extension theorem [11, Th. 0.2.13].

3.3. How the energies $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu})$ and $\kappa(R\boldsymbol{\mu}, R\boldsymbol{\mu})$ are related to each other?

In accordance with an electrostatic interpretation of a condenser $\mathbf{A}$, we assume that the law of interaction between the charges lying on the plates A_i, $i \in I$, is determined by the matrix $(\alpha_i\alpha_j)_{i,j\in I}$. Then the *mutual energy* of $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathfrak{M}^+(\mathbf{A})$ is given by the expression[2]

$$\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}_1) := \sum_{i,j\in I} \alpha_i\alpha_j\kappa(\mu^i, \mu_1^j). \tag{3.2}$$

For $\boldsymbol{\mu} = \boldsymbol{\mu}_1$ the mutual energy defines the *energy* $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu})$ of $\boldsymbol{\mu}$. Let $\mathcal{E}^+(\mathbf{A})$ consist of all $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ with $-\infty < \kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) < \infty$.

Lemma 3.5. *For $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ to have finite energy, it is necessary and sufficient that $\mu^i \in \mathcal{E}$ for all $i \in I$ and $\sum_{i\in I} \|\mu^i\|^2 < \infty$.*

Proof. This follows immediately from the above definitions due to the inequality $2\kappa(\nu_1, \nu_2) \leqslant \|\nu_1\|^2 + \|\nu_2\|^2$ for ν_1, $\nu_2 \in \mathcal{E}$. □

In view of the convexity of $\mathfrak{M}^+(\mathbf{A})$, Lemma 3.5 yields that also $\mathcal{E}^+(\mathbf{A})$ forms a *convex cone*.

In order to establish relations between the mutual energies of vector measures and those of their (scalar) R-images, we need the following two lemmas, the former being well known (see, e.g., [12]). In both, Y is a locally compact Hausdorff space.

[2]It will be shown below (see Corollary 3.10) that the mutual energy is well defined and finite (hence, the series in (3.2) converges absolutely) at least for all measures from $\mathcal{E}^+(\mathbf{A})$.

Lemma 3.6. *If $\psi \in \Phi(\mathrm{Y})$ is given, then the map $\nu \mapsto \langle \psi, \nu \rangle$ is vaguely lower semicontinuous on $\mathfrak{M}^+(\mathrm{Y})$.*

In particular, this implies that the potential $\kappa(\cdot, \nu)$ of any $\nu \in \mathfrak{M}^+(\mathrm{X})$ belongs to $\Phi(\mathrm{X})$.

Lemma 3.7. *Consider an (L^+, L^-)-condenser $\mathbf{B} = (B_\ell)_{\ell \in L}$ in Y, a vector measure $\boldsymbol{\omega} = (\omega^\ell)_{\ell \in L} \in \mathfrak{M}^+(\mathbf{B})$, and a function $\psi \in \Phi(\mathrm{Y})$. For $\langle \psi, R\boldsymbol{\omega} \rangle$ to be finite, it is necessary and sufficient that $\sum_{\ell \in L} \alpha_\ell \langle \psi, \omega^\ell \rangle$ converge absolutely, and then*

$$\langle \psi, R\boldsymbol{\omega} \rangle = \sum_{\ell \in L} \alpha_\ell \langle \psi, \mu^\ell \rangle.$$

Proof. We can assume ψ to be nonnegative, for if not, we replace ψ by a function $\psi' \geqslant 0$ obtained by adding to ψ a suitable constant $c > 0$, which is always possible since a lower semicontinuous function is bounded from below on a compact space. Hence,

$$\langle \psi, R\boldsymbol{\omega}^+ \rangle \geqslant \sum_{\ell \in L^+,\ \ell \leqslant N} \langle \psi, \omega^\ell \rangle \quad \text{for all} \quad N \in L^+.$$

On the other hand, the sum of ω^ℓ over all $\ell \in L^+$ that do not exceed N approaches $R\boldsymbol{\omega}^+$ vaguely as $N \to \infty$; consequently, by Lemma 3.6,

$$\langle \psi, R\boldsymbol{\omega}^+ \rangle \leqslant \lim_{N \to \infty} \sum_{\ell \in L^+,\ \ell \leqslant N} \langle \psi, \omega^\ell \rangle.$$

Combining the last two inequalities and then letting $N \to \infty$, we get

$$\langle \psi, R\boldsymbol{\omega}^+ \rangle = \sum_{\ell \in L^+} \langle \psi, \omega^\ell \rangle.$$

Since the same holds true for $R\boldsymbol{\omega}^-$ and L^- instead of $R\boldsymbol{\omega}^+$ and L^+, the lemma follows. $\square$

To apply Lemma 3.7 to the condenser $\mathbf{A} \times \mathbf{A} := (A_i \times A_j)_{(i,j) \in I \times I}$ in $\mathrm{X} \times \mathrm{X}$ with $\alpha_{(i,j)} := \alpha_i \alpha_j$, we observe that any $\boldsymbol{\omega} \in \mathfrak{M}^+(\mathbf{A} \times \mathbf{A})$ can be written as $\boldsymbol{\mu} \otimes \boldsymbol{\mu}_1 := (\mu^i \otimes \mu_1^j)_{(i,j) \in I \times I}$, where $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathfrak{M}^+(\mathbf{A})$. Therefore,

$$R(\boldsymbol{\mu} \otimes \boldsymbol{\mu}_1) = \sum_{i,j \in I} \alpha_i \alpha_j \mu^i \otimes \mu_1^j = R\boldsymbol{\mu} \otimes R\boldsymbol{\mu}_1.$$

If, moreover, $\psi = \kappa \in \Phi(\mathrm{X} \times \mathrm{X})$, then we arrive at the following assertion.

Corollary 3.8. *Given $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathfrak{M}^+(\mathbf{A})$, we $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}_1) = \kappa(R\boldsymbol{\mu}, R\boldsymbol{\mu}_1)$, the identity being understood in the sense that each of its sides is finite whenever so is the other and then they coincide.*

Hence, $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ belongs to $\mathcal{E}^+(\mathbf{A})$ if and only if $R\boldsymbol{\mu} \in \mathcal{E}$ and, furthermore,

$$\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) = \kappa(R\boldsymbol{\mu}, R\boldsymbol{\mu}) \quad \text{for all} \quad \boldsymbol{\mu} \in \mathcal{E}^+(\mathbf{A}). \tag{3.3}$$

In view of the positive definiteness of the kernel, this yields the following property of positivity of the energy $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu})$, which was not obvious a priori.

Corollary 3.9. *For all $\boldsymbol{\mu} \in \mathcal{E}^+(\mathbf{A})$, it is true that $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) \geqslant 0$.*

Corollary 3.10. *For any $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathcal{E}^+(\mathbf{A})$, we have*

$$\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}_1) = \kappa(R\boldsymbol{\mu}, R\boldsymbol{\mu}_1) = \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu_1^j), \tag{3.4}$$

and the series here converges absolutely.

Proof. For any $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathcal{E}^+(\mathbf{A})$, we get $R\boldsymbol{\mu}, R\boldsymbol{\mu}_1 \in \mathcal{E}$; hence, $\kappa(R\boldsymbol{\mu}, R\boldsymbol{\mu}_1)$ is finite. Therefore, repeated application of Corollary 3.8 gives the desired conclusion. □

3.4. Semimetric space of vector measures of finite energy

Theorem 3.11. *$\mathcal{E}^+(\mathbf{A})$ forms a semimetric space with the semimetric $\|\cdot\|_{\mathcal{E}^+(\mathbf{A})}$, defined by (1.2), and this space is isometric to its R-image. The semimetric $\|\cdot\|_{\mathcal{E}^+(\mathbf{A})}$ is a metric if and only if the kernel κ is strictly positive definite while all A_i, $i \in I$, are mutually disjoint.*

Proof. Fix $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathcal{E}^+(\mathbf{A})$. Applying Corollary 3.10 to $\kappa(R\boldsymbol{\mu}_k, R\boldsymbol{\mu}_t)$, $k, t = 1, 2$, we get

$$\|R\boldsymbol{\mu}_1 - R\boldsymbol{\mu}_2\|^2_{\mathcal{E}} = \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu_1^i - \mu_2^i, \mu_1^j - \mu_2^j),$$

where the series converges absolutely. Compared with (1.2), this relation yields

$$\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{E}^+(\mathbf{A})} = \|R\boldsymbol{\mu}_1 - R\boldsymbol{\mu}_2\|_{\mathcal{E}}. \tag{3.5}$$

Since $\|\cdot\|_{\mathcal{E}}$ is a seminorm on $\mathcal{E}$, the theorem follows. □

From now on, $\mathcal{E}^+(\mathbf{A})$ will always be treated as a semimetric space with the semimetric $\|\cdot\| := \|\cdot\|_{\mathcal{E}^+(\mathbf{A})}$. Since $\mathcal{E}^+(\mathbf{A})$ and its R-image are isometric, similar to the terminology in $\mathcal{E}$ we shall call the topology on $\mathcal{E}^+(\mathbf{A})$ *strong*.

Two elements of $\mathcal{E}^+(\mathbf{A})$, $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, are said to be *equivalent in* $\mathcal{E}^+(\mathbf{A})$ if $\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\| = 0$. Observe that the equivalence in $\mathcal{E}^+(\mathbf{A})$ implies R-equivalence (i.e., then $R\boldsymbol{\mu}_1 = R\boldsymbol{\mu}_2$) provided the kernel κ is strictly positive definite, and it implies the identity (i.e., then $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$) if, moreover, all A_i, $i \in I$, are mutually disjoint.

4. Constrained minimal f-weighted energy problem

4.1. Statement of the problem

Consider an external field $\mathbf{f} = (f_i)_{i \in I}$ satisfying Case I or Case II (see the Introduction), and assume each f_i to affect the charges on A_i only. The **f**-*weighted energy* $G_{\mathbf{f}}(\boldsymbol{\mu})$ of $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ is defined by (1.1), and let $\mathcal{E}^+_{\mathbf{f}}(\mathbf{A})$ consist of all $\boldsymbol{\mu} \in \mathcal{E}^+(\mathbf{A})$ with $-\infty < G_{\mathbf{f}}(\boldsymbol{\mu}) < \infty$.

Also fix a nonnegative vector measure $\boldsymbol{\sigma} \in \mathfrak{M}^+(\mathbf{A})$, called a *constraint associated with* $\mathbf{A}$, a numerical vector $\mathbf{a} = (a_i)_{i \in I}$ with $a_i > 0$, and a vector-valued function $\mathbf{g} = (g_i)_{i \in I}$, where all the $g_i : \mathrm{X} \to (0, \infty)$ are continuous. We define

$$\mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}) := \bigl\{\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A}) : \ \boldsymbol{\mu} \leqslant \boldsymbol{\sigma}\bigr\},$$

where $\boldsymbol{\mu} \leqslant \boldsymbol{\sigma}$ means that $\sigma^i - \mu^i \geqslant 0$ for all $i \in I$, and

$$\mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \bigl\{\boldsymbol{\mu} \in \mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}) : \ \langle g_i, \mu^i \rangle = a_i \quad \text{for all} \ \ i \in I\bigr\},$$
$$\mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \cap \mathcal{E}^+(\mathbf{A}),$$
$$\mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \cap \mathcal{E}^+_{\mathbf{f}}(\mathbf{A})$$

and then we introduce the extremal value

$$G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \inf_{\mu \in \mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})} G_{\mathbf{f}}(\boldsymbol{\mu}). \tag{4.1}$$

In (4.1), as usual, the infimum over the empty set is taken to be $+\infty$.

If $\mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty or, which is equivalent, if it is true that[3]

$$G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty, \tag{4.2}$$

then the following problem makes sense.

Problem. Does there exist $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}} \in \mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with $G_{\mathbf{f}}(\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}) = G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$?

Along with its electrostatic interpretation, such a problem has found various important applications in approximation theory (see, e.g., [8, 9, 21]). The problem is called *solvable* if the class $\mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ of all the minimizers $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$ is nonempty.

4.2. On the uniqueness of minimizers

Lemma 4.1. *If $\boldsymbol{\lambda}$ and $\widehat{\boldsymbol{\lambda}}$ belong to $\mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, then*

$$\|\boldsymbol{\lambda} - \widehat{\boldsymbol{\lambda}}\|_{\mathcal{E}^+(\mathbf{A})} = 0. \tag{4.3}$$

Proof. It follows from the convexity of $\mathcal{E}^+(\mathbf{A})$ (see Sec. 3.3) that so is $\mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, which makes it possible to conclude from (1.1), (3.3), and (4.1) that

$$4G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \leqslant 4G_{\mathbf{f}}\Bigl(\frac{\boldsymbol{\lambda} + \widehat{\boldsymbol{\lambda}}}{2}\Bigr) = \|R\boldsymbol{\lambda} + R\widehat{\boldsymbol{\lambda}}\|^2 + 4\langle \mathbf{f}, \boldsymbol{\lambda} + \widehat{\boldsymbol{\lambda}}\rangle.$$

On the other hand, applying the parallelogram identity in the pre-Hilbert space $\mathcal{E}$ to $R\boldsymbol{\lambda}$ and $R\widehat{\boldsymbol{\lambda}}$ and then adding and subtracting $4\langle \mathbf{f}, \boldsymbol{\lambda} + \widehat{\boldsymbol{\lambda}}\rangle$, we get

$$\|R\boldsymbol{\lambda} - R\widehat{\boldsymbol{\lambda}}\|^2 = -\|R\boldsymbol{\lambda} + R\widehat{\boldsymbol{\lambda}}\|^2 - 4\langle \mathbf{f}, \boldsymbol{\lambda} + \widehat{\boldsymbol{\lambda}}\rangle + 2G_{\mathbf{f}}(\boldsymbol{\lambda}) + 2G_{\mathbf{f}}(\widehat{\boldsymbol{\lambda}}).$$

When combined with the preceding relation, this yields

$$0 \leqslant \|R\boldsymbol{\lambda} - R\widehat{\boldsymbol{\lambda}}\|^2 \leqslant -4G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) + 2G_{\mathbf{f}}(\boldsymbol{\lambda}) + 2G_{\mathbf{f}}(\widehat{\boldsymbol{\lambda}}) = 0,$$

which establishes (4.3) because of (3.5). $\square$

Thus, any two minimizers (if exist) are equivalent in $\mathcal{E}^+(\mathbf{A})$. Consequently, they are R-equivalent if the kernel κ is strictly positive definite, and they are equal if, moreover, all A_i, $i \in I$, are mutually disjoint.

[3] See Lemma 5.5 below for necessary and (or) sufficient conditions for (4.2) to hold. Then, actually, $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ has to be finite (see Corollary 5.4).

5. Elementary properties of $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$

Before analyzing the existence of minimizers and their continuity, we provide some auxiliary results, to be needed in the sequel. Write

$$g_{i,\inf} := \inf_{x\in A_i} g_i(x), \qquad g_{i,\sup} := \sup_{x\in A_i} g_i(x).$$

5.1. Monotonicity of $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$

On the collection of all (I^+,I^-)-condensers in X, it is natural to introduce an ordering relation $\leqslant$ by declaring $\mathbf{A}'\leqslant\mathbf{A}$ to mean that $A_i'\subset A_i$ for all $i\in I$. Here, $\mathbf{A}'=(A_i')_{i\in I}$. If now $\boldsymbol{\sigma}$ is a constraint associated with $\mathbf{A}$ and $\boldsymbol{\sigma}'$ is that associated with $\mathbf{A}'$, then we write $(\mathbf{A}',\boldsymbol{\sigma}')\leqslant(\mathbf{A},\boldsymbol{\sigma})$ provided $\mathbf{A}'\leqslant\mathbf{A}$ and $\boldsymbol{\sigma}'\leqslant\boldsymbol{\sigma}$. Then $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is a nonincreasing function of $(\mathbf{A},\boldsymbol{\sigma})$, namely

$$G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})\leqslant G^{\boldsymbol{\sigma}'}_{\mathbf{f}}(\mathbf{A}',\mathbf{a},\mathbf{g}) \quad \text{whenever} \quad (\mathbf{A}',\boldsymbol{\sigma}')\leqslant(\mathbf{A},\boldsymbol{\sigma}). \tag{5.1}$$

We shall employ the technique of exhaustion of $\mathbf{A}$ by compact $\mathbf{K}$. In doing so, we shall need the following notation and elementary lemma.

Given $\mathbf{A}$, let $\{\mathbf{K}\}_{\mathbf{A}}$ stand for the increasing family of all compact condensers $\mathbf{K}=(K_i)_{i\in I}$ such that $\mathbf{K}\leqslant\mathbf{A}$. For any $\boldsymbol{\mu}\in\mathfrak{M}^+(\mathbf{A})$ and $\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}$, let $\mu^i_{\mathbf{K}}$ denote the trace of μ^i upon K_i, i.e. $\mu^i_{\mathbf{K}}:=\mu^i_{K_i}$, and let $\boldsymbol{\mu}_{\mathbf{K}}:=(\mu^i_{\mathbf{K}})_{i\in I}$. Observe that, if $\boldsymbol{\sigma}$ is a constraint associated with $\mathbf{A}$, then $\boldsymbol{\sigma}_{\mathbf{K}}=(\sigma^i_{\mathbf{K}})_{i\in I}$ is that associated with $\mathbf{K}$. We further write $\widehat{\boldsymbol{\mu}}_{\mathbf{K}}:=(\hat{\mu}^i_{\mathbf{K}})_{i\in I}$, where

$$\hat{\mu}^i_{\mathbf{K}}:=\frac{a_i}{\langle g_i,\mu^i_{\mathbf{K}}\rangle}\,\mu^i_{\mathbf{K}}. \tag{5.2}$$

Lemma 5.1. *Fix $\boldsymbol{\mu}\in\mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. For every $\varepsilon>0$, there exists $\mathbf{K}_0\in\{\mathbf{K}\}_{\mathbf{A}}$ such that, for all $\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}$ that follow $\mathbf{K}_0$,*

$$\widehat{\boldsymbol{\mu}}_{\mathbf{K}}\in\mathcal{E}^+_{(1+\varepsilon)\boldsymbol{\sigma}_{\mathbf{K}},\mathbf{f}}(\mathbf{K},\mathbf{a},\mathbf{g}). \tag{5.3}$$

Proof. Application of [12, Lemma 1.2.2] yields

$$\langle g_i,\mu^i\rangle=\lim_{\mathbf{K}\uparrow\mathbf{A}}\langle g_i,\mu^i_{\mathbf{K}}\rangle, \qquad i\in I, \tag{5.4}$$

$$\langle f_i,\mu^i\rangle=\lim_{\mathbf{K}\uparrow\mathbf{A}}\langle f_i,\mu^i_{\mathbf{K}}\rangle, \qquad i\in I, \tag{5.5}$$

$$\kappa(\mu^i,\mu^j)=\lim_{\mathbf{K}\uparrow\mathbf{A}}\kappa(\mu^i_{\mathbf{K}},\mu^j_{\mathbf{K}}), \quad i,j\in I. \tag{5.6}$$

Fix $\varepsilon>0$. By (5.4)–(5.6), for every $i\in I$ one can choose a compact set $K_i^0\subset A_i$ so that, for all compact sets K_i with the property $K_i^0\subset K_i\subset A_i$,

$$\frac{a_i}{\langle g_i,\mu^i_{K_i}\rangle}<1+\varepsilon\, i^{-2}, \tag{5.7}$$

$$\bigl|\langle f_i,\mu^i\rangle-\langle f_i,\mu^i_{K_i}\rangle\bigr|<\varepsilon\, i^{-2}, \tag{5.8}$$

$$\bigl|\|\mu^i\|^2-\|\mu^i_{K_i}\|^2\bigr|<\varepsilon^2 i^{-4}. \tag{5.9}$$

Having denoted $\mathbf{K}_0 := (K_i^0)_{i\in I}$, for every $\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}$ that follows $\mathbf{K}_0$ we get

$$\widehat{\boldsymbol{\mu}}_{\mathbf{K}} \in \mathcal{E}^+_{(1+\varepsilon)\boldsymbol{\sigma}_{\mathbf{K}}}(\mathbf{K}, \mathbf{a}, \mathbf{g}),$$

the finiteness of the energy being obtained from (5.2), (5.7), and (5.9) with help of Lemma 3.5. Furthermore, since $\sum_{i\in I} \langle f_i, \mu^i\rangle$ converges absolutely, we conclude from (5.7) and (5.8) that so does $\sum_{i\in I} \langle f_i, \hat{\mu}^i_{\mathbf{K}}\rangle$. This means (5.3) as claimed. □

5.2. It is true that $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) > -\infty$

To prove the estimate, announced in the title, we need the following two lemmas.

Lemma 5.2. *Let Case* II *take place, i.e., let $f_i = \alpha_i \kappa(\cdot, \zeta)$ for all $i \in I$, where $\zeta \in \mathcal{E}$ is given. Then the classes $\mathcal{E}^+(\mathbf{A})$ and $\mathcal{E}^+_{\mathbf{f}}(\mathbf{A})$ coincide and, furthermore,*

$$G_{\mathbf{f}}(\boldsymbol{\mu}) = \|R\boldsymbol{\mu} + \zeta\|^2 - \|\zeta\|^2 \quad \text{for all} \ \ \boldsymbol{\mu} \in \mathcal{E}^+(\mathbf{A}). \tag{5.10}$$

Proof. Applying Lemma 3.7 to $\boldsymbol{\mu} \in \mathcal{E}^+(\mathbf{A})$ and each of $\kappa(\cdot, \zeta^+)$ and $\kappa(\cdot, \zeta^-)$, we get

$$\langle \mathbf{f}, \boldsymbol{\mu}\rangle = \sum_{i\in I} \alpha_i \int \kappa(x, \zeta)\, d\mu^i(x) = \kappa(\zeta, R\boldsymbol{\mu}), \tag{5.11}$$

where the series converges absolutely. Hence, $\boldsymbol{\mu} \in \mathcal{E}^+_{\mathbf{f}}(\mathbf{A})$. Now, substituting (3.3) and (5.11) into (1.1) gives (5.10) as required. □

Lemma 5.3. *Consider a condenser $\mathbf{B} = (B_\ell)_{\ell\in L}$ in a locally compact space* Y*, $\mathbf{u} = (u_\ell)_{\ell\in L}$ with $u_\ell \in \Phi(\mathrm{Y})$, and $\mathfrak{F} \subset \mathfrak{M}^+(\mathbf{B})$ with the property that*

$$\sup_{\boldsymbol{\omega}\in\mathfrak{F}} \omega^\ell(\mathrm{Y}) < \infty \quad \text{for all} \ \ \ell \in L \tag{5.12}$$

unless Y *is noncompact. Then $\langle \mathbf{u}, \boldsymbol{\omega}\rangle$ is well defined for all $\boldsymbol{\omega} \in \mathfrak{F}$, and*

$$-\infty < \inf_{\boldsymbol{\omega}\in\mathfrak{F}} \langle \mathbf{u}, \boldsymbol{\omega}\rangle \leqslant \infty.$$

Proof. We can assume Y to be compact, for if not, then $u_\ell \geqslant 0$ for all $\ell \in L$ and the lemma is obvious. But then $\mathbf{B}$ is to be finite while every u_ℓ, being lower semicontinuous, is bounded from below by $-c_\ell$, where $0 < c_\ell < \infty$. Hence, by (5.12),

$$-\infty < -c_\ell \sup_{\boldsymbol{\omega}\in\mathfrak{F}} \omega^\ell(\mathrm{Y}) \leqslant \langle u_\ell, \omega^\ell\rangle \leqslant \infty,$$

which in view of the finiteness of L yields the lemma. □

Corollary 5.4. $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) > -\infty$.

Proof. We can consider Case I, since otherwise the corollary follows from (5.10). Then $f_i \in \Phi(\mathrm{X})$ for all $i \in I$. Furthermore, if X is compact, then $g_{i,\inf} > 0$ and

$$\sup_{\boldsymbol{\mu}\in\mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})} \mu^i(\mathrm{X}) \leqslant a_i g^{-1}_{i,\inf} < \infty.$$

By Lemma 5.3,

$$-\infty < M_0 \leqslant \langle \mathbf{f}, \boldsymbol{\mu}\rangle \leqslant \infty \quad \text{for all} \ \ \boldsymbol{\mu} \in \mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}), \tag{5.13}$$

which together with Corollary 3.9 completes the proof. □

5.3. When does $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})<\infty$ hold?

Let $C(E)=C_\kappa(E)$ denote the interior capacity of a set $E\subset \mathrm{X}$ relative to the kernel κ (see [12]).

The following assertion provides necessary and (or) sufficient conditions for relation (4.2) to hold (or, which is equivalent, for $\mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ to be nonempty).

Lemma 5.5. *If* (4.2) *is true, then necessarily*

$$C\bigl(\{x\in A_i:\ |f_i(x)|<\infty\}\bigr)>0\quad\textit{for all}\ \ i\in I.$$

In the case where

$$\sum_{i\in I} a_i\, g_{i,\inf}^{-1}<\infty, \tag{5.14}$$

for (4.2) *to hold, it is sufficient that the following conditions be both satisfied:*

(a) *for every* $\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}$, $\boldsymbol{\sigma}_{\mathbf{K}}$ *has finite energy;*

(b) *there exists* $M\in(0,\infty)$ *not depending on* i *and such that* $\langle g_i,\sigma^i_{A_i^M}\rangle>a_i$, *where* $A_i^M:=\{x\in A_i:\ |f_i(x)|\leqslant M\}$, $i\in I$.

Proof. To prove the necessity part of the lemma, fix $\boldsymbol{\mu}\in\mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$; then, by Lemma 5.1, $\widehat{\boldsymbol{\mu}}_{\mathbf{K}}$ has finite $\mathbf{f}$-weighted energy provided $\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}$ is sufficiently large. Suppose, contrary to our claim, that $C\bigl(\{x\in A_{i_0}:\ |f_{i_0}(x)|<\infty\}\bigr)=0$ for some $i_0\in I$. Since $\hat{\mu}^{i_0}_{\mathbf{K}}$ has finite energy and is compactly supported in A_{i_0}, [12, Lemma 2.3.1] shows that $|f_{i_0}(x)|=\infty$ holds $\hat{\mu}^{i_0}_{\mathbf{K}}$-almost everywhere ($\hat{\mu}^{i_0}_{\mathbf{K}}$-a.e.) in X. This is impossible, for $\hat{\mu}^{i_0}_{\mathbf{K}}$ is nonzero while $\langle\mathbf{f},\widehat{\boldsymbol{\mu}}_{\mathbf{K}}\rangle$ is finite.

To establish the sufficient part, suppose (5.14), (a), and (b) to be satisfied. Then for every $i\in I$ one can choose a compact set $K_i\subset A_i^M$ so that

$$\langle g_i,\sigma^i_{K_i}\rangle>a_i, \tag{5.15}$$

which is seen from (b) due to [12, Lemma 1.2.2]. Having denoted $\mathbf{K}:=(K_i)_{i\in I}$, we consider the vector measure $\widehat{\boldsymbol{\sigma}}_{\mathbf{K}}$ with the components $\hat{\sigma}^i_{\mathbf{K}}$, defined by (5.2) with $\sigma^i_{\mathbf{K}}$ in place of $\mu^i_{\mathbf{K}}$. It follows from (5.15) that $\widehat{\boldsymbol{\sigma}}_{\mathbf{K}}\in\mathcal{E}^+_{\boldsymbol{\sigma}_{\mathbf{K}}}(\mathbf{K},\mathbf{a},\mathbf{g})$, the finiteness of the energy being obtained from (a) in view of Lemma 3.5. Furthermore, since

$$\sum_{i\in I}\langle|f_i|,\hat{\sigma}^i_{\mathbf{K}}\rangle\leqslant M\sum_{i\in I}\frac{a_i\sigma^i_{\mathbf{K}}(\mathrm{X})}{\langle g_i,\sigma^i_{\mathbf{K}}\rangle}\leqslant M\sum_{i\in I}a_i\,g_{i,\inf}^{-1},$$

we actually have $\widehat{\boldsymbol{\sigma}}_{\mathbf{K}}\in\mathcal{E}^+_{\boldsymbol{\sigma}_{\mathbf{K}},\mathbf{f}}(\mathbf{K},\mathbf{a},\mathbf{g})$ by (5.14), and so $G^{\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K},\mathbf{a},\mathbf{g})<\infty$. Since $(\mathbf{K},\boldsymbol{\sigma}_{\mathbf{K}})\leqslant(\mathbf{A},\boldsymbol{\sigma})$, this together with (5.1) yields (4.2) as was to be proved. $\square$

Remark 5.6. If $\mathbf{A}$ is finite, then Lemma 5.5 remains true with (b) replaced by the following assumption: for every $i\in I$, $\langle g_i,\sigma^i\rangle>a_i$ while $|f_i|\neq\infty$ locally σ^i-a.e. (see [25, Lemma 4]).

6. Main results

From now on, (4.2) is always assumed to hold. Observe that, according to Corollary 5.4, $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is then actually finite.

Suppose for a moment that the condenser $\mathbf{A}$ is compact. Then the class $\mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is vaguely bounded and closed and hence, by Lemma 3.2, it is vaguely compact. If, moreover, $\mathbf{A}$ is finite, κ is continuous on $A^+ \times A^-$, and Case I holds, then $G_{\mathbf{f}}(\boldsymbol{\mu})$ is vaguely lower semicontinuous on $\mathcal{E}^+_{\mathbf{f}}(\mathbf{A})$ and, therefore, the existence of minimizers $\lambda^{\boldsymbol{\sigma}}_{\mathbf{A}}$ immediately follows (cf. [14, 19, 20, 22]).

However, these arguments break down if any of the above-mentioned four assumptions is dropped, and then the problem on the existence of minimizers $\lambda^{\boldsymbol{\sigma}}_{\mathbf{A}}$ becomes rather nontrivial. In particular, $\mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is no longer vaguely compact if any of the A_i is noncompact. Another difficulty is that $G_{\mathbf{f}}(\boldsymbol{\mu})$ might not be vaguely lower semicontinuous on $\mathcal{E}^+_{\mathbf{f}}(\mathbf{A})$ when Case II takes place.

To solve the problem on the existence of minimizers $\lambda^{\boldsymbol{\sigma}}_{\mathbf{A}}$ in the general case, we develop an approach based on both the vague and strong topologies in the semimetric space $\mathcal{E}^+(\mathbf{A})$, introduced for vector measures of finite dimensions in [25, 26, 28]. For $I = \{1\}$, see also [27] (compare with [8, 9, 21]).

6.1. Standing assumptions

In addition to (4.2), in all that follows it is always required that the kernel κ is consistent and either $I^- = \varnothing$, or there hold (5.14) and the following condition:

$$\sup_{x \in A^+,\ y \in A^-} \kappa(x, y) < \infty. \tag{6.1}$$

Remark 6.1. Note that these assumptions on a kernel are not too restrictive. In particular, they all are satisfied by the Newtonian, Riesz, or Green kernels in $\mathbb{R}^n$, $n \geqslant 2$, provided the Euclidean distance between A^+ and A^- is nonzero, as well as by the restriction of the logarithmic kernel in $\mathbb{R}^2$ to the open unit disk.

6.2. Minimizers: existence and vague compactness

A proposition $u(x)$ involving a variable point $x \in \mathrm{X}$ is said to subsist *nearly everywhere* (n.e.) in E, where $E \subset \mathrm{X}$, if the set of all $x \in E$ for which u fails to hold is of interior capacity zero.

Theorem 6.2. *Under the standing assumptions, suppose, moreover, for every $i \in I$ the following* (a)–(c) *to hold:*

(a) *Either $g_{i,\inf} > 0$ or A_i can be written as a countable union of compact sets;*

(b) *Either $g_{i,\sup} < \infty$ or there exist $r_i \in (1, \infty)$ and $\tau_i \in \mathcal{E}$ with the property*

$$g_i^{r_i}(x) \leqslant \kappa(x, \tau_i) \quad \text{n.e. in} \ \ A_i; \tag{6.2}$$

(c) *A_i either is compact or has finite interior capacity.*

Then, for any $\boldsymbol{\sigma}$, $\mathbf{f}$, and $\mathbf{a}$, $\mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty and vaguely compact.

Remark 6.3. If I^- is nonempty, then condition (a) follows immediately from (5.14) and, hence, it can be omitted. It also holds automatically if the space X is countable at infinity (e.g., for $\mathrm{X} = \mathbb{R}^n$).

Remark 6.4. Regarding condition (c), note that a compact set $K \subset \mathrm{X}$ might be of infinite capacity; $C(K)$ is necessarily finite provided the kernel is strictly positive definite [12]. On the other hand, even for the Newtonian kernel, sets of finite capacity might be noncompact [17].

Remark 6.5. Condition (c) is essential for the validity of Theorem 6.2. See Sec. 12 for some examples, illustrating its sharpness.

Corollary 6.6. *If* $\mathbf{A} = \mathbf{K}$ *is compact, then, for any* $\boldsymbol{\sigma}$, $\mathbf{f}$, $\mathbf{g}$, *and* $\mathbf{a}$, $\mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *is nonempty and vaguely compact.*

Proof. This is an immediate consequence of Theorem 6.2, since g_i is bounded on K_i. □

6.3. On continuity of $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$ with respect to $(\mathbf{A}, \boldsymbol{\sigma})$

We write $\mathbf{A}_s \downarrow \mathbf{A}$, where $\mathbf{A}_s = (A_i^s)_{i\in I}$, $s \in S$, is a net of condensers, if $\mathbf{A}_{s_2} \leqslant \mathbf{A}_{s_1}$ whenever $s_1 \leqslant s_2$ and

$$\bigcap_{s\in S} A_i^s = A_i \quad \text{for all} \;\; i \in I.$$

Theorem 6.7. *Let* $\mathbf{A}_s \downarrow \mathbf{A}$, *and let for some* $s_0 \in S$ *all the assumptions*[4] *of Theorem* 6.2 *with* $\mathbf{A}_{s_0}$ *instead of* $\mathbf{A}$ *be satisfied. Let* $\boldsymbol{\sigma}_s$ *be a constraint associated with* $\mathbf{A}_s$, *and let* $(\boldsymbol{\sigma}_s)_{s\in S}$ *decrease and converge vaguely to* $\boldsymbol{\sigma}$. *Then*

$$G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \lim_{s\in S} G^{\boldsymbol{\sigma}_s}_{\mathbf{f}}(\mathbf{A}_s, \mathbf{a}, \mathbf{g}).$$

Fix arbitrary $\boldsymbol{\lambda}^{\boldsymbol{\sigma}_s}_{\mathbf{A}_s} \in \mathfrak{S}^{\boldsymbol{\sigma}_s}_{\mathbf{f}}(\mathbf{A}_s, \mathbf{a}, \mathbf{g})$, *where* $s \geqslant s_0$, *and* $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *— such minimizers exist. Then every vague cluster point of the net* $(\boldsymbol{\lambda}^{\boldsymbol{\sigma}_s}_{\mathbf{A}_s})_{s\in S}$ *is an element of* $\mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. *Furthermore,* $\boldsymbol{\lambda}^{\boldsymbol{\sigma}_s}_{\mathbf{A}_s} \to \boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$ *strongly, i.e.*

$$\lim_{s\in S} \|\boldsymbol{\lambda}^{\boldsymbol{\sigma}_s}_{\mathbf{A}_s} - \boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}\|_{\mathcal{E}^+(\mathbf{A}_{s_0})} = 0.$$

Corollary 6.8. *Under the assumptions of Theorem* 6.7, *if, moreover, the kernel* κ *is strictly positive definite* (*hence, perfect*) *and all* $A_i^{s_0}$, $i \in I$, *are mutually disjoint, then the* (*unique*) *minimizer* $\boldsymbol{\lambda}^{\boldsymbol{\sigma}_s}_{\mathbf{A}_s} \in \mathfrak{S}^{\boldsymbol{\sigma}_s}_{\mathbf{f}}(\mathbf{A}_s, \mathbf{a}, \mathbf{g})$, *where* $s \geqslant s_0$, *approaches the* (*unique*) *minimizer* $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *both vaguely and strongly.*

In the rest of Sec. 6.3, let $\mathbf{A}$ and $\mathbf{g}$ satisfy all the conditions of Theorem 6.2. We proceed by analyzing continuity properties of $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$ under exhaustion of $\mathbf{A}$ by compact $\mathbf{K}$.

[4] Including the standing ones.

Theorem 6.9. *There exists a net* $(\beta^*_{\mathbf{K}})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}} \subset (1,\infty)$ *decreasing to* 1 *and such that, for any* $\beta_{\mathbf{K}} \in [1, \beta^*_{\mathbf{K}}]$,

$$G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) = \lim_{\mathbf{K}\uparrow\mathbf{A}} G^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K},\mathbf{a},\mathbf{g}), \tag{6.3}$$

where $\boldsymbol{\sigma}_{\mathbf{K}} := (\sigma^i_{K_i})_{i\in I}$. *Fix arbitrary* $\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}} \in \mathfrak{S}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K},\mathbf{a},\mathbf{g})$, *where* $\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}$ *is sufficiently large, and* $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *— such minimizers exist. Then every vague cluster point of the net* $(\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$ *is an element of* $\mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. *Furthermore,* $\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}} \to \boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$ *strongly, i.e.*

$$\lim_{\mathbf{K}\uparrow\mathbf{A}} \|\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}} - \boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}\|_{\mathcal{E}^+(\mathbf{A})} = 0.$$

Corollary 6.10. *With the notation of Theorem* 6.9, *if the kernel* κ *is strictly positive definite* (*hence, perfect*) *and all* A_i, $i\in I$, *are mutually disjoint, then the* (*unique*) *minimizer* $\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}} \in \mathfrak{S}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K},\mathbf{a},\mathbf{g})$, *where* $\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}$, *approaches the* (*unique*) *minimizer* $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *both vaguely and strongly.*

Remark 6.11. The value $G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ remains unchanged if $\mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ in its definition is replaced by the class of all $\boldsymbol{\mu}\in\mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ such that $\operatorname{supp}\mu^i$, $i\in I$, are *compact*. Indeed, this is concluded from (6.3) with $\beta_{\mathbf{K}}=1$ for all $\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}$.

The proofs of Theorems 6.2, 6.7 and 6.9, to be given in Sections 9, 10 and 11 below (see also Sec. 8 for some crucial auxiliary notions and results), are based on a theorem on the strong completeness of proper subspaces of the semimetric space $\mathcal{E}^+(\mathbf{A})$, which is a subject of Sec. 7.

7. Strong completeness of classes of vector measures

Recall that we are working under the standing assumptions, stated in Sec. 6.1. Write

$$\mathfrak{M}^+(\mathbf{A},\leqslant\mathbf{a},\mathbf{g}) := \bigl\{\boldsymbol{\mu}\in\mathfrak{M}^+(\mathbf{A}): \quad \langle g_i,\mu^i\rangle \leqslant a_i \ \text{ for all } \ i\in I\bigr\},$$

$$\mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A},\leqslant\mathbf{a},\mathbf{g}) := \mathfrak{M}^+(\mathbf{A},\leqslant\mathbf{a},\mathbf{g}) \cap \mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}),$$

$$\mathcal{E}^+(\mathbf{A},\leqslant\mathbf{a},\mathbf{g}) := \mathfrak{M}^+(\mathbf{A},\leqslant\mathbf{a},\mathbf{g}) \cap \mathcal{E}^+(\mathbf{A}),$$

and

$$\mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A},\leqslant\mathbf{a},\mathbf{g}) := \mathcal{E}^+(\mathbf{A},\leqslant\mathbf{a},\mathbf{g}) \cap \mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}).$$

Our next purpose is to show that $\mathcal{E}^+(\mathbf{A},\leqslant\mathbf{a},\mathbf{g})$ and $\mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A},\leqslant\mathbf{a},\mathbf{g})$, treated as topological subspaces of the semimetric space $\mathcal{E}^+(\mathbf{A})$, are *strongly complete*.

7.1. Auxiliary assertions

Lemma 7.1. *$\mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ and $\mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ are vaguely bounded and, hence, they are vaguely compact.*

Proof. Fix $i \in I$, and let a compact set $K_i \subset A_i$ be given. Since g_i is positive and continuous, the relation

$$a_i \geqslant \langle g_i, \mu^i \rangle \geqslant \mu^i(K_i) \min_{x \in K_i} g_i(x), \quad \text{where} \ \ \mu \in \mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g}),$$

yields

$$\sup_{\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})} \mu^i(K_i) < \infty.$$

This implies that $\mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is vaguely bounded; hence, by Lemma 3.2, it is vaguely relatively compact. In fact, it is vaguely compact, since it is vaguely closed in consequence of Lemma 3.6 with $\mathrm{Y} = A_i$ and $\psi = g_i$. Having observed that also $\mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A})$ is vaguely closed, we then conclude that $\mathfrak{M}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is vaguely compact as well, which completes the proof. □

Lemma 7.2. *If a net $(\boldsymbol{\mu}_s)_{s \in S} \subset \mathcal{E}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is strongly bounded, then every its vague cluster point $\boldsymbol{\mu}$ has finite energy.*

Proof. Note that, by (3.3), the net of scalar measures $(R\boldsymbol{\mu}_s)_{s \in S} \subset \mathcal{E}$ is strongly bounded as well. We proceed by showing that so are $(R\boldsymbol{\mu}_s^+)_{s \in S}$ and $(R\boldsymbol{\mu}_s^-)_{s \in S}$, i.e.,

$$\sup_{s \in S} \|R\boldsymbol{\mu}_s^{\pm}\|^2 < \infty. \tag{7.1}$$

Of course, this needs to be verified only when $I^- \neq \varnothing$; then, according to the standing assumptions, (5.14) and (6.1) hold. Since $\langle g_i, \mu_s^i \rangle \leqslant a_i$, we conclude that $\mu_s^i(\mathrm{X}) \leqslant a_i g_{i,\inf}^{-1}$ for all $i \in I$ and $s \in S$. Therefore, by (5.14),

$$\sup_{s \in S} R\boldsymbol{\mu}_s^{\pm}(\mathrm{X}) \leqslant \sum_{i \in I} a_i g_{i,\inf}^{-1} < \infty.$$

Because of (6.1), this implies that $\kappa(R\boldsymbol{\mu}_s^+, R\boldsymbol{\mu}_s^-)$ remains bounded from above on S; hence, so do $\|R\boldsymbol{\mu}_s^+\|^2$ and $\|R\boldsymbol{\mu}_s^-\|^2$.

If $(\boldsymbol{\mu}_d)_{d \in D}$ is a subnet of $(\boldsymbol{\mu}_s)_{s \in S}$ that converges vaguely to $\boldsymbol{\mu}$, then $(R\boldsymbol{\mu}_d^+)_{d \in D}$ and $(R\boldsymbol{\mu}_d^-)_{d \in D}$ converge vaguely to $R\boldsymbol{\mu}^+$ and $R\boldsymbol{\mu}^-$, respectively. Using the fact that the map $(\nu_1, \nu_2) \mapsto \nu_1 \otimes \nu_2$ from $\mathfrak{M}^+(\mathrm{X}) \times \mathfrak{M}^+(\mathrm{X})$ into $\mathfrak{M}^+(\mathrm{X} \times \mathrm{X})$ is vaguely continuous (see [2, Chap. 3, § 5, exerc. 5]) and applying Lemma 3.6 to $\mathrm{Y} = \mathrm{X} \times \mathrm{X}$ and $\psi = \kappa$, we conclude from (7.1) that $R\boldsymbol{\mu}^+$ and $R\boldsymbol{\mu}^-$ are both of finite energy. By Corollary 3.8, this means $\boldsymbol{\mu} \in \mathcal{E}^+(\mathbf{A})$, as was to be proved. □

Corollary 7.3. *If a net $(\boldsymbol{\mu}_s)_{s \in S} \subset \mathcal{E}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is strongly bounded, then*

$$\sup_{s \in S} \|\mu_s^i\|^2 < \infty, \quad i \in I. \tag{7.2}$$

Proof. As an application of Lemma 5.3, we obtain

$$\inf_{s\in S} \sum_{i\neq j,\ i,j\in I^{\pm}} \langle \kappa, \mu_s^i \otimes \mu_s^j \rangle > -\infty.$$

When combined with (3.4) and (7.1), this yields the corollary. □

7.2. Strong completeness of $\mathcal{E}^+(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$ and $\mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$

Theorem 7.4. *The following assertions hold:*

(i) *The semimetric space $\mathcal{E}^+(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$ is complete. In more detail, if $(\boldsymbol{\mu}_s)_{s\in S}$ is a strong Cauchy net in $\mathcal{E}^+(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$ and $\boldsymbol{\mu}$ is one of its vague cluster points (such a $\boldsymbol{\mu}$ exists), then $\boldsymbol{\mu} \in \mathcal{E}^+(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$ and $\boldsymbol{\mu}_s \to \boldsymbol{\mu}$ strongly, i.e.*

$$\lim_{s\in S} \|\boldsymbol{\mu}_s - \boldsymbol{\mu}\|_{\mathcal{E}^+(\mathbf{A})} = 0. \tag{7.3}$$

(ii) *If the kernel κ is strictly positive definite while all A_i, $i \in I$, are mutually disjoint, then the strong topology on $\mathcal{E}^+(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$ is finer than the vague one. In more detail, if $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathcal{E}^+(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$ converges strongly to $\boldsymbol{\mu}_0 \in \mathcal{E}^+(\mathbf{A})$, then actually $\boldsymbol{\mu}_0 \in \mathcal{E}^+(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$ and $\boldsymbol{\mu}_s \to \boldsymbol{\mu}_0$ vaguely.*

(iii) *Both the assertions* (i) *and* (ii) *remain valid if $\mathcal{E}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is replaced everywhere by $\mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$.*

Proof. To verify (i), fix a strong Cauchy net $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathcal{E}^+(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g})$. Since such a net converges strongly to any of its strong cluster points, $(\boldsymbol{\mu}_s)_{s\in S}$ can be assumed to be strongly bounded. Then, by Lemmas 7.1 and 7.2, a vague cluster point $\boldsymbol{\mu}$ of $(\boldsymbol{\mu}_s)_{s\in S}$ exists and, moreover,

$$\boldsymbol{\mu} \in \mathcal{E}^+(\mathbf{A}, \leqslant\mathbf{a}, \mathbf{g}). \tag{7.4}$$

We next proceed by proving (7.3). Of course, there is no loss of generality in assuming $(\boldsymbol{\mu}_s)_{s\in S}$ to converge vaguely to $\boldsymbol{\mu}$. Then, by Lemma 3.3, $(R\boldsymbol{\mu}_s^+)_{s\in S}$ and $(R\boldsymbol{\mu}_s^-)_{s\in S}$ converge vaguely to $R\boldsymbol{\mu}^+$ and $R\boldsymbol{\mu}^-$, respectively. Since, by (7.1), these nets are strongly bounded in $\mathcal{E}^+$, the property (C_2) (see Sec. 2) shows that they approach $R\boldsymbol{\mu}^+$ and $R\boldsymbol{\mu}^-$, respectively, in the weak topology as well, and so $R\boldsymbol{\mu}_s \to R\boldsymbol{\mu}$ weakly. This gives, by (3.5),

$$\|\boldsymbol{\mu}_s - \boldsymbol{\mu}\|^2 = \|R\boldsymbol{\mu}_s - R\boldsymbol{\mu}\|^2 = \lim_{\ell\in S} \kappa(R\boldsymbol{\mu}_s - R\boldsymbol{\mu}, R\boldsymbol{\mu}_s - R\boldsymbol{\mu}_\ell)$$

and hence, by the Cauchy–Schwarz inequality,

$$\|\boldsymbol{\mu}_s - \boldsymbol{\mu}\|^2 \leqslant \|\boldsymbol{\mu}_s - \boldsymbol{\mu}\| \liminf_{\ell\in S} \|\boldsymbol{\mu}_s - \boldsymbol{\mu}_\ell\|,$$

which proves (7.3) as required, because $\|\boldsymbol{\mu}_s - \boldsymbol{\mu}_\ell\|$ becomes arbitrarily small when $s, \ell \in S$ are sufficiently large. The proof of (i) is complete.

To establish (ii), suppose now that the kernel κ is strictly positive definite, while all A_i, $i \in I$, are mutually disjoint, and let the net $(\boldsymbol{\mu}_s)_{s\in S}$ converge strongly to some $\boldsymbol{\mu}_0 \in \mathcal{E}^+(\mathbf{A})$. Given a vague limit point $\boldsymbol{\mu}$ of $(\boldsymbol{\mu}_s)_{s\in S}$, we conclude from (7.3) that $\|\boldsymbol{\mu}_0 - \boldsymbol{\mu}\| = 0$, hence $R\boldsymbol{\mu}_0 = R\boldsymbol{\mu}$ since κ is strictly positive definite, and finally $\boldsymbol{\mu}_0 = \boldsymbol{\mu}$ because A_i, $i \in I$, are mutually disjoint. In view of (7.4), this

means that $\boldsymbol{\mu}_0 \in \mathcal{E}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$, which is a part of the desired conclusion. Moreover, $\boldsymbol{\mu}_0$ has thus been shown to be identical to any vague cluster point of $(\boldsymbol{\mu}_s)_{s\in S}$. Since the vague topology is Hausdorff, this implies that $\boldsymbol{\mu}_0$ is actually the vague limit of $(\boldsymbol{\mu}_s)_{s\in S}$ (cf. [1, Chap. I, § 9, n° 1, cor.]), as claimed.

Finally, we observe that all the arguments applied above remain valid if $\mathcal{E}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is replaced everywhere by $\mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$. This yields (iii). □

Remark 7.5. Since the semimetric spaces $\mathcal{E}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ and $\mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ are isometric to their R-images, Theorem 7.4 has thus singled out *strongly complete* topological subspaces of the pre-Hilbert space $\mathcal{E}$, whose elements are *signed* measures. This is of a independent interest because, according to a well-known counterexample by H. Cartan [5], the whole space $\mathcal{E}$ is strongly incomplete even for the Newtonian kernel $|x-y|^{2-n}$ in $\mathbb{R}^n$, $n \geqslant 3$.

8. Extremal measures in the constrained energy problem

To apply Theorem 7.4 to the constrained energy problem, we next proceed by introducing the concept of extremal measure defined as a strong and, simultaneously, a vague limit point of a minimizing net. See below for strict definitions and related auxiliary results.

8.1. Extremal measures: existence, uniqueness, and vague compactness

Definition 8.1. We call a net $(\boldsymbol{\mu}_s)_{s\in S}$ *minimizing* if $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and

$$\lim_{s\in S} G_{\mathbf{f}}(\boldsymbol{\mu}_s) = G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{8.1}$$

Let $\mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ consist of all minimizing nets, and let $\mathcal{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ be the union of the vague cluster sets of $(\boldsymbol{\mu}_s)_{s\in S}$, where $(\boldsymbol{\mu}_s)_{s\in S}$ ranges over $\mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$.

Definition 8.2. We call $\boldsymbol{\gamma} \in \mathcal{E}^+(\mathbf{A})$ *extremal* if there exists $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ that converges to $\boldsymbol{\gamma}$ both strongly and vaguely; such a net $(\boldsymbol{\mu}_s)_{s\in S}$ is said to *generate* $\boldsymbol{\gamma}$. The class of all extremal measures will be denoted by $\mathfrak{E}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$.

Lemma 8.3. *The following assertions hold true:*

(i) *From every minimizing net one can select a subnet generating an extremal measure; hence,* $\mathfrak{E}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *is nonempty. Furthermore,*

$$\mathfrak{E}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g}) \tag{8.2}$$

and

$$\mathfrak{E}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \mathcal{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{8.3}$$

(ii) *Every minimizing net converges strongly to every extremal measure; hence,* $\mathfrak{E}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *is contained in an equivalence class in* $\mathcal{E}^+(\mathbf{A})$.

(iii) *The class* $\mathfrak{E}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *is vaguely compact.*

Proof. Fix $(\boldsymbol{\mu}_s)_{s\in S}$ and $(\boldsymbol{\nu}_t)_{t\in T}$ in $\mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. Then

$$\lim_{(s,t)\in S\times T} \|\boldsymbol{\mu}_s-\boldsymbol{\nu}_t\|^2=0, \tag{8.4}$$

where $S\times T$ is the directed product (see, e.g., [16, Chap. 2, § 3]) of the directed sets S and T. Indeed, in the same manner as in the proof of Lemma 4.1 we get

$$0\leqslant \|R\boldsymbol{\mu}_s-R\boldsymbol{\nu}_t\|^2\leqslant -4G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})+2G_{\mathbf{f}}(\boldsymbol{\mu}_s)+2G_{\mathbf{f}}(\boldsymbol{\nu}_t),$$

which yields (8.4) when combined with (8.1).

Relation (8.4) implies that $(\boldsymbol{\mu}_s)_{s\in S}$ is strongly fundamental. Therefore, by Theorem 7.4, (iii), for every vague cluster point $\boldsymbol{\mu}$ of $(\boldsymbol{\mu}_s)_{s\in S}$ (such a $\boldsymbol{\mu}$ exists) we have $\boldsymbol{\mu}\in\mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A},\leqslant\mathbf{a},\mathbf{g})$ and $\boldsymbol{\mu}_s\to\boldsymbol{\mu}$ strongly. Thus, $\boldsymbol{\mu}$ is an extremal measure; actually,

$$\mathcal{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})\subset\mathfrak{C}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}).$$

Since the inverse inclusion is obvious, relations (8.2) and (8.3) follow.

Having thus proved (i), we proceed by verifying (ii). Fix arbitrary $(\boldsymbol{\mu}_s)_{s\in S}\in\mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ and $\boldsymbol{\gamma}\in\mathfrak{C}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. Then, according to Definition 8.2, there exists a net in $\mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$, say $(\boldsymbol{\nu}_t)_{t\in T}$, that converges to $\boldsymbol{\gamma}$ strongly. Repeated application of (8.4) shows that also $(\boldsymbol{\mu}_s)_{s\in S}$ converges to $\boldsymbol{\gamma}$ strongly, as claimed.

To establish (iii), it is enough to prove that $\mathcal{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is vaguely compact. Fix $(\boldsymbol{\gamma}_s)_{s\in S}\subset\mathcal{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. It follows from (8.2) and Lemma 7.1 that there exists a vague cluster point $\boldsymbol{\gamma}_0$ of $(\boldsymbol{\gamma}_s)_{s\in S}$; let $(\boldsymbol{\gamma}_t)_{t\in T}$ be a subnet of $(\boldsymbol{\gamma}_s)_{s\in S}$ that converges vaguely to $\boldsymbol{\gamma}_0$. Then for every $t\in T$ one can choose $(\boldsymbol{\mu}_{s_t})_{s_t\in S_t}\in\mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ converging vaguely to $\boldsymbol{\gamma}_t$. Consider the Cartesian product $\prod\{S_t: t\in T\}$ — that is, the collection of all functions β on T with $\beta(t)\in S_t$, and let D denote the directed product $T\times\prod\{S_t: t\in T\}$. Given $(t,\beta)\in D$, write $\boldsymbol{\mu}_{(t,\beta)}:=\boldsymbol{\mu}_{\beta(t)}$. Then the theorem on iterated limits from [16, Chap. 2, § 4] yields that the net $(\boldsymbol{\mu}_{(t,\beta)})_{(t,\beta)\in D}$ belongs to $\mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ and converges vaguely to $\boldsymbol{\gamma}_0$. Thus, $\boldsymbol{\gamma}_0\in\mathcal{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. □

Corollary 8.4. *If Case II takes place, then*

$$G_{\mathbf{f}}(\boldsymbol{\gamma})=G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})\quad\textit{for all}\ \ \boldsymbol{\gamma}\in\mathfrak{C}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}). \tag{8.5}$$

Proof. Applying (5.10) to $\boldsymbol{\mu}_s$, $s\in S$, and $\boldsymbol{\gamma}$, where $(\boldsymbol{\mu}_s)_{s\in S}\in\mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ and $\boldsymbol{\gamma}\in\mathfrak{C}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ are arbitrarily given, and using the fact that, in accordance with Lemma 8.3, $\boldsymbol{\mu}_s\to\boldsymbol{\gamma}$ strongly, we get

$$G_{\mathbf{f}}(\boldsymbol{\gamma})=\|R\boldsymbol{\gamma}+\zeta\|^2-\|\zeta\|^2=\lim_{s\in S}\big[\|R\boldsymbol{\mu}_s+\zeta\|^2-\|\zeta\|^2\big]=\lim_{s\in S}G_{\mathbf{f}}(\boldsymbol{\mu}_s).$$

Substituting (8.1) into the last relation gives (8.5), as desired. □

8.2. Extremal measures: g_i-masses of the i-components

Lemma 8.5. *Let* κ, $\mathbf{A}$, $\mathbf{a}$, *and* $\mathbf{g}$ *satisfy all the assumptions* (a)–(c) *of Theorem* 6.2, *and let a net* $(\boldsymbol{\mu}_s)_{s\in S}\subset\mathcal{E}^+(\mathbf{A})$ *be strongly bounded and converge vaguely to* $\boldsymbol{\mu}_0$. *If, moreover,* $\langle g_i,\mu^i_s\rangle=a_i$ *for all* $s\in S$ *and* $i\in I$, *then*

$$\langle g_i,\mu^i_0\rangle=a_i\quad\textit{for all}\ \ i\in I. \tag{8.6}$$

Proof. Fix $i \in I$. By Corollary 7.3, the net $(\mu_s^i)_{s\in S}$ is strongly bounded as well.

Also note that A_i can be written as a countable union of μ_s^i-integrable sets, where $s \in S$ is given. Indeed, this is obvious if A_i is a countable union of compact sets; otherwise, due to condition (a) of Theorem 6.2, we have $g_{i,\inf} > 0$ and hence $\mu_s^i(A_i) \leqslant a_i g_{i,\inf}^{-1} < \infty$. Therefore, the concept of local μ_s^i-negligibility and that of μ_s^i-negligibility coincide. Together with [12, Lemma 2.3.1], this yields that any proposition holds μ_s^i-a.e. in X provided it holds n.e. in A_i.

We proceed by establishing (8.6). Of course, this needs to be done only if the set A_i is noncompact; then, by condition (c), its capacity has to be finite. Hence, by [12, Th. 4.1], for every $E \subset A_i$ there exists a measure $\theta_E \in \mathcal{E}^+(\overline{E})$, called an *interior equilibrium measure* associated with E, which admits the properties

$$\theta_E(\mathrm{X}) = \|\theta_E\|^2 = C(E), \tag{8.7}$$

$$\kappa(x, \theta_E) \geqslant 1 \quad \text{n.e. in} \quad E. \tag{8.8}$$

Also observe that there is no loss of generality in assuming g_i to satisfy (6.2) with some $r_i \in (1, \infty)$ and $\tau_i \in \mathcal{E}$. Indeed, otherwise, due to condition (b) of Theorem 6.2, g_i has to be bounded from above (say by M), which combined with (8.8) again gives (6.2) for $\tau_i := M^{r_i}\,\theta_{A_i}$, $r_i \in (1, \infty)$ being arbitrary.

We treat A_i as a locally compact space with the topology induced from X. Given $E \subset A_i$, let χ_E denote its characteristic function and let $E^c := A_i \setminus E$. Further, let $\{K_i\}$ be the increasing family of all compact subsets K_i of A_i. Since $g_i\chi_{K_i}$ is upper semicontinuous on A_i while $(\mu_s^i)_{s\in S}$ converges to μ_0^i vaguely, from Lemma 3.6 we get

$$\langle g_i\chi_{K_i}, \mu_0^i\rangle \geqslant \limsup_{s\in S}\, \langle g_i\chi_{K_i}, \mu_s^i\rangle \quad \text{for every} \quad K_i \in \{K_i\}.$$

On the other hand, application of Lemma 1.2.2 from [12] yields

$$\langle g_i, \mu_0^i\rangle = \lim_{K_i\in\{K_i\}} \langle g_i\chi_{K_i}, \mu_0^i\rangle.$$

Combining the last two relations, we obtain

$$a_i \geqslant \langle g_i, \mu_0^i\rangle \geqslant \limsup_{(s,K_i)\in S\times\{K_i\}} \langle g_i\chi_{K_i}, \mu_s^i\rangle = a_i - \liminf_{(s,K_i)\in S\times\{K_i\}} \langle g_i\chi_{K_i^c}, \mu_s^i\rangle,$$

$S \times \{K_i\}$ being the directed product of the directed sets S and $\{K_i\}$. Hence, if we prove

$$\liminf_{(s,K_i)\in S\times\{K_i\}} \langle g_i\chi_{K_i^c}, \mu_s^i\rangle = 0, \tag{8.9}$$

the desired relation (8.6) follows.

Consider an interior equilibrium measure $\theta_{K_i^c}$, where $K_i \in \{K_i\}$ is given. Then application of Lemma 4.1.1 and Theorem 4.1 from [12] shows that

$$\|\theta_{K_i^c} - \theta_{\tilde K_i^c}\|^2 \leqslant \|\theta_{K_i^c}\|^2 - \|\theta_{\tilde K_i^c}\|^2 \quad \text{provided} \quad K_i \subset \tilde K_i.$$

Furthermore, it is clear from (8.7) that the net $\|\theta_{K_i^c}\|$, $K_i \in \{K_i\}$, is bounded and nonincreasing, and hence fundamental in $\mathbb{R}$. The preceding inequality thus yields that the net $(\theta_{K_i^c})_{K_i\in\{K_i\}}$ is strongly fundamental in $\mathcal{E}$. Since, clearly, it converges

vaguely to zero, the property (C_1) (see. Sec. 2) implies that zero is also one of its strong limits and, hence,

$$\lim_{K_i\in\{K_i\}} \|\theta_{K_i^c}\| = 0. \tag{8.10}$$

Write $q_i := r_i(r_i-1)^{-1}$, where $r_i \in (1,\infty)$ is the number involved in condition (6.2). Combining (6.2) with (8.8) shows that the inequality

$$g_i(x)\chi_{K_i^c}(x) \leqslant \kappa(x,\tau_i)^{1/r_i}\kappa(x,\theta_{K_i^c})^{1/q_i}$$

subsists n.e. in A_i and, hence, μ_s^i-a.e. in X. Having integrated this relation with respect to μ_s^i, we then apply the Hölder and, subsequently, the Cauchy–Schwarz inequalities to the integrals on the right. This gives

$$\begin{aligned}\langle g_i\chi_{K_i^c},\mu_s^i\rangle &\leqslant \Bigl[\int\kappa(x,\tau_i)\,d\mu_s^i(x)\Bigr]^{1/r_i}\Bigl[\int\kappa(x,\theta_{K_i^c})\,d\mu_s^i(x)\Bigr]^{1/q_i}\\ &\leqslant \|\tau_i\|^{1/r_i}\|\theta_{K_i^c}\|^{1/q_i}\|\mu_s^i\|.\end{aligned}$$

Taking limits here along $S\times\{K\}$ and using (7.2) and (8.10), we obtain (8.9) as desired. □

Corollary 8.6. *Under the assumptions of Theorem* 6.2*, we have*

$$\mathfrak{C}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}) \subset \mathcal{E}_{\boldsymbol{\sigma}}^+(\mathbf{A},\mathbf{a},\mathbf{g}). \tag{8.11}$$

Proof. Fix $\boldsymbol{\gamma}\in\mathfrak{C}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$; then there exists a net $(\boldsymbol{\mu}_s)_{s\in S}\subset\mathcal{E}_{\boldsymbol{\sigma},\mathbf{f}}^+(\mathbf{A},\mathbf{a},\mathbf{g})$ converging to $\boldsymbol{\gamma}$ strongly and vaguely. Taking a subnet if necessary, we assume $(\boldsymbol{\mu}_s)_{s\in S}$ to be strongly bounded. Then, by Lemma 8.5, $\langle g_i,\gamma^i\rangle = a_i$ for all $i\in I$, which together with (8.2) gives (8.11). □

9. Proof of Theorem 6.2

Fix an extremal measure $\boldsymbol{\gamma}\in\mathfrak{C}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$ — it exists by Lemma 8.3, (i), and choose a net $(\boldsymbol{\mu}_s)_{s\in S}\in\mathbb{M}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$ that converges to $\boldsymbol{\gamma}$ both strongly and vaguely. We are going to show that $\boldsymbol{\gamma}$ is a minimizer, i.e.

$$\boldsymbol{\gamma}\in\mathfrak{S}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}). \tag{9.1}$$

According to Corollary 8.6, we have $\boldsymbol{\gamma}\in\mathcal{E}_{\boldsymbol{\sigma}}^+(\mathbf{A},\mathbf{a},\mathbf{g})$. Hence, (9.1) will be established once we prove that $\sum_{i\in I}\langle f_i,\gamma^i\rangle$ converges absolutely, so that

$$\boldsymbol{\gamma}\in\mathcal{E}_{\boldsymbol{\sigma},\mathbf{f}}^+(\mathbf{A},\mathbf{a},\mathbf{g}), \tag{9.2}$$

and

$$G_{\mathbf{f}}(\boldsymbol{\gamma}) \leqslant G_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}). \tag{9.3}$$

To this end, assume Case I to hold, since otherwise (9.2) and (9.3) have already been established by Lemma 5.2 and Corollary 8.4, respectively. Then, in consequence of Lemma 5.3 (see (5.13) with $\boldsymbol{\mu}=\boldsymbol{\gamma}$), $\langle\mathbf{f},\boldsymbol{\gamma}\rangle$ is well defined and

$$\langle\mathbf{f},\boldsymbol{\gamma}\rangle > -\infty. \tag{9.4}$$

Besides, from the strong and vague convergence of $(\boldsymbol{\mu}_s)_{s\in S}$ to $\boldsymbol{\gamma}$ we obtain

$$G_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}) = \lim_{s\in S}\big[\|\boldsymbol{\mu}_s\|^2 + 2\langle\mathbf{f},\boldsymbol{\mu}_s\rangle\big] = \|\boldsymbol{\gamma}\|^2 + 2\lim_{s\in S}\langle\mathbf{f},\boldsymbol{\mu}_s\rangle \tag{9.5}$$

(consequently, $\lim_{s\in S}\langle\mathbf{f},\boldsymbol{\mu}_s\rangle$ exists and is finite) and

$$\begin{aligned}\langle\mathbf{f},\boldsymbol{\gamma}\rangle = \sum_{i\in I}\langle f_i,\gamma^i\rangle &\leqslant \sum_{i\in I}\liminf_{s\in S}\langle f_i,\mu_s^i\rangle \\ &\leqslant \lim_{s\in S}\sum_{i\in I}\langle f_i,\mu_s^i\rangle = \lim_{s\in S}\langle\mathbf{f},\boldsymbol{\mu}_s\rangle < \infty.\end{aligned} \tag{9.6}$$

Combining (9.4) and (9.6) proves (9.2), while substituting (9.6) into (9.5) gives (9.3), and the required inclusion (9.1) follows.

It has thus been proved that $\mathfrak{E}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}) \subset \mathfrak{S}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$. This inclusion can certainly be inverted, since any minimizer $\boldsymbol{\lambda}_{\mathbf{A}}^{\boldsymbol{\sigma}}$ can be thought as an extremal measure generated by the constant net $(\boldsymbol{\lambda}_{\mathbf{A}}^{\boldsymbol{\sigma}})$. On account of (8.3), we get

$$\mathfrak{S}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}) = \mathfrak{E}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}) = \mathcal{M}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}).$$

Therefore Lemma 8.3 shows that $\mathfrak{S}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is vaguely compact. The proof is complete. □

10. Proof of Theorem 6.7

It is seen from (4.2), (5.1) and Corollary 5.4 that, under the assumptions of the theorem, $G_{\mathbf{f}}^{\boldsymbol{\sigma}_s}(\mathbf{A}_s,\mathbf{a},\mathbf{g})$ increases as s ranges through S and

$$-\infty < \lim_{s\in S} G_{\mathbf{f}}^{\boldsymbol{\sigma}_s}(\mathbf{A}_s,\mathbf{a},\mathbf{g}) \leqslant G_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}) < \infty.$$

Besides, in accordance with Theorem 6.2, for every $s \geqslant s_0$ there is a minimizer $\boldsymbol{\lambda}_s := \boldsymbol{\lambda}_{\mathbf{A}_s}^{\boldsymbol{\sigma}_s} \in \mathfrak{S}_{\mathbf{f}}^{\boldsymbol{\sigma}_s}(\mathbf{A}_s,\mathbf{a},\mathbf{g})$. Therefore, $\lim_{s\in S} G_{\mathbf{f}}(\boldsymbol{\lambda}_s)$ exists and

$$-\infty < \lim_{s\in S} G_{\mathbf{f}}(\boldsymbol{\lambda}_s) \leqslant G_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}) < \infty. \tag{10.1}$$

Also observe that, since $\mathbf{A}_s$ and $\boldsymbol{\sigma}_s$ decrease along S, it is true that

$$\boldsymbol{\lambda}_s \in \mathcal{E}_{\boldsymbol{\sigma}_\ell,\mathbf{f}}^+(\mathbf{A}_\ell,\mathbf{a},\mathbf{g}) \quad \text{for all} \quad s \geqslant \ell \geqslant s_0.$$

We proceed by showing that

$$\|\boldsymbol{\lambda}_{s_2} - \boldsymbol{\lambda}_{s_1}\|^2 \leqslant G_{\mathbf{f}}(\boldsymbol{\lambda}_{s_2}) - G_{\mathbf{f}}(\boldsymbol{\lambda}_{s_1}) \tag{10.2}$$

whenever $s_0 \leqslant s_1 \leqslant s_2$. For every $t \in (0,1]$, $\boldsymbol{\mu} := (1-t)\boldsymbol{\lambda}_{s_1} + t\boldsymbol{\lambda}_{s_2}$ belongs to the class $\mathcal{E}_{\boldsymbol{\sigma}_{s_1},\mathbf{f}}^+(\mathbf{A}_{s_1},\mathbf{a},\mathbf{g})$, and therefore $G_{\mathbf{f}}(\boldsymbol{\mu}) \geqslant G_{\mathbf{f}}(\boldsymbol{\lambda}_{s_1})$. Evaluating the left-hand side of this inequality and then letting $t \to 0$, we get

$$-\|\boldsymbol{\lambda}_{s_1}\|^2 + \kappa(\boldsymbol{\lambda}_{s_1},\boldsymbol{\lambda}_{s_2}) - \langle\mathbf{f},\boldsymbol{\lambda}_{s_1}\rangle + \langle\mathbf{f},\boldsymbol{\lambda}_{s_2}\rangle \geqslant 0,$$

and (10.2) follows.

Due to (10.1), the net $G_{\mathbf{f}}(\boldsymbol{\lambda}_s)$, $s \in S$, is fundamental in $\mathbb{R}$. When combined with (10.2), this implies that $\boldsymbol{\lambda}_s$, $s \geqslant \ell \geqslant s_0$, form a fundamental net

in $\mathcal{E}^+_{\boldsymbol{\sigma}_\ell}(\mathbf{A}_\ell, \mathbf{a}, \mathbf{g})$. Hence, by Theorem 7.4, there exists a vague cluster point $\boldsymbol{\lambda}$ of $(\boldsymbol{\lambda}_s)_{s\in S}$ and the following two assertions hold:

$$\text{(i)}\quad \boldsymbol{\lambda} \in \mathcal{E}^+_{\boldsymbol{\sigma}_s}(\mathbf{A}_s, \leqslant\mathbf{a}, \mathbf{g}) \quad \text{for all } s \geqslant s_0; \qquad \text{(ii)}\quad \boldsymbol{\lambda}_s \to \boldsymbol{\lambda} \text{ strongly.}$$

However, Lemma 8.5 with $\mathbf{A}_s$ instead of $\mathbf{A}$ shows that assertion (i) can be strengthened as follows: $\boldsymbol{\lambda} \in \mathcal{E}^+_{\boldsymbol{\sigma}_s}(\mathbf{A}_s, \mathbf{a}, \mathbf{g})$ for all $s \geqslant s_0$. In turn, this implies that, actually, $\boldsymbol{\lambda} \in \mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, since $\boldsymbol{\sigma}_s \to \boldsymbol{\sigma}$ vaguely while $\mathbf{A}_s \downarrow \mathbf{A}$.

What has already been established yields that the proof of the theorem will be complete once we show that $\sum_{i\in I} \langle f_i, \lambda^i\rangle$ converges absolutely, so that

$$\boldsymbol{\lambda} \in \mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}), \tag{10.3}$$

and

$$\langle \mathbf{f}, \boldsymbol{\lambda}\rangle \leqslant \lim_{s\in S} \langle \mathbf{f}, \boldsymbol{\lambda}_s\rangle. \tag{10.4}$$

Note that $\lim_{s\in S} \langle \mathbf{f}, \boldsymbol{\lambda}_s\rangle$ exists and is finite, which is clear from (10.1) and (ii).

We can suppose Case I to hold, since otherwise (10.3) is already known from Lemma 5.2 while (10.4) can be obtained directly from (5.11) and assertion (ii). Therefore, by (5.13) with $\boldsymbol{\mu} = \boldsymbol{\lambda}$, $\langle \mathbf{f}, \boldsymbol{\lambda}\rangle$ is well defined and $\langle \mathbf{f}, \boldsymbol{\lambda}\rangle > -\infty$. Taking a subnet if necessary, we can also assume that $\boldsymbol{\lambda}_s \to \boldsymbol{\lambda}$ vaguely. Then,

$$-\infty < \langle \mathbf{f}, \boldsymbol{\lambda}\rangle = \sum_{i\in I} \langle f_i, \lambda^i\rangle \leqslant \sum_{i\in I} \liminf_{s\in S} \langle f_i, \lambda^i_s\rangle \leqslant \lim_{s\in S} \sum_{i\in I} \langle f_i, \lambda^i_s\rangle < \infty,$$

and the required relations (10.3) and (10.4) follow. □

11. Proof of Theorem 6.9

We begin by establishing the relation

$$G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \lim_{\mathbf{K}\uparrow\mathbf{A}} G^{\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K}, \mathbf{a}, \mathbf{g}). \tag{11.1}$$

For every $\boldsymbol{\mu} \in \mathcal{E}^+_{\boldsymbol{\sigma},\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, consider $\widehat{\boldsymbol{\mu}}_{\mathbf{K}} = (\hat{\mu}^i_{\mathbf{K}})_{i\in I}$ defined by (5.2). Fix an arbitrary $\varepsilon > 0$ and choose $\mathbf{K}_0$ so that for all $\mathbf{K}$ that follow $\mathbf{K}_0$ inclusion (5.3) holds. This yields

$$G_{\mathbf{f}}(\widehat{\boldsymbol{\mu}}_{\mathbf{K}}) \geqslant G^{(1+\varepsilon)\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K}, \mathbf{a}, \mathbf{g}). \tag{11.2}$$

We next proceed by showing that

$$G_{\mathbf{f}}(\boldsymbol{\mu}) = \lim_{\mathbf{K}\uparrow\mathbf{A}} G_{\mathbf{f}}(\widehat{\boldsymbol{\mu}}_{\mathbf{K}}). \tag{11.3}$$

To this end, it can be assumed that $\kappa \geqslant 0$; for if not, then $\mathbf{A}$ must be finite since X is compact, and (11.3) follows from (5.4)–(5.6). Therefore, for all $\mathbf{K} \geqslant \mathbf{K}_0$ and $i \in I$ we get

$$\|\mu^i_{\mathbf{K}}\| \leqslant \|\mu^i\| \leqslant \|R\boldsymbol{\mu}^+ + R\boldsymbol{\mu}^-\|, \tag{11.4}$$

$$\|\mu^i - \mu^i_{\mathbf{K}}\| < \varepsilon\, i^{-2}, \tag{11.5}$$

the latter being clear from (5.9) because of $\kappa(\mu^i_{\mathbf{K}}, \mu^i - \mu^i_{\mathbf{K}}) \geqslant 0$. Also observe that

$$\bigl|\|\boldsymbol{\mu}\|^2 - \|\widehat{\boldsymbol{\mu}}_{\mathbf{K}}\|^2\bigr| \leqslant \sum_{i,j\in I} \Bigl|\kappa(\mu^i, \mu^j) - \frac{a_i}{\langle g_i, \mu^i_{\mathbf{K}}\rangle}\frac{a_j}{\langle g_j, \mu^j_{\mathbf{K}}\rangle}\,\kappa(\mu^i_{\mathbf{K}}, \mu^j_{\mathbf{K}})\Bigr|$$

$$\leqslant \sum_{i,j\in I}\Bigl[\kappa(\mu^i - \mu^i_{\mathbf{K}}, \mu^j) + \kappa(\mu^i_{\mathbf{K}}, \mu^j - \mu^j_{\mathbf{K}}) + \Bigl(\frac{a_i}{\langle g_i, \mu^i_{\mathbf{K}}\rangle}\frac{a_j}{\langle g_j, \mu^j_{\mathbf{K}}\rangle} - 1\Bigr)\,\kappa(\mu^i_{\mathbf{K}}, \mu^j_{\mathbf{K}})\Bigr].$$

When combined with (5.7), (5.8), (11.4), and (11.5), this yields

$$\bigl|G_{\mathbf{f}}(\boldsymbol{\mu}) - G_{\mathbf{f}}(\widehat{\boldsymbol{\mu}}_{\mathbf{K}})\bigr| \leqslant M\varepsilon \quad \text{for all } \ \mathbf{K} \geqslant \mathbf{K}_0,$$

where M is finite and independent of $\mathbf{K}$, and the required relation (11.3) follows.

Substituting (11.2) into (11.3), in view of (5.1) and the arbitrary choice of $\boldsymbol{\mu}$ we get

$$G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \geqslant \lim_{\mathbf{K}\uparrow\mathbf{A}} G^{(1+\varepsilon)\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K}, \mathbf{a}, \mathbf{g}) \geqslant G^{(1+\varepsilon)\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}).$$

Letting $\varepsilon \to 0$ and applying Theorem 6.7 to both $\mathbf{A}$ and $\mathbf{K}$, we obtain

$$G^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \lim_{\varepsilon\to 0}\Bigl[\lim_{\mathbf{K}\uparrow\mathbf{A}} G^{(1+\varepsilon)\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K}, \mathbf{a}, \mathbf{g})\Bigr] = \lim_{\mathbf{K}\uparrow\mathbf{A}} G^{\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K}, \mathbf{a}, \mathbf{g}),$$

which proves (11.1) as desired.

Fix $\boldsymbol{\lambda}^{\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}} \in \mathfrak{S}^{\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K}, \mathbf{a}, \mathbf{g})$, where $\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}$. As follows from (11.1), the net $(\boldsymbol{\lambda}^{\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$ belongs to $\mathbb{M}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and, hence, it is strongly fundamental.

Further, for every $\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}$ choose $\beta^*_{\mathbf{K}} \in (1, \infty)$ such that $(\beta^*_{\mathbf{K}})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$ decreases to 1 and, for all $\beta_{\mathbf{K}} \in [1, \beta^*_{\mathbf{K}}]$ and $\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}} \in \mathfrak{S}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{f}}(\mathbf{K}, \mathbf{a}, \mathbf{g})$,

$$\lim_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}} \|\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}} - \boldsymbol{\lambda}^{\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}}\| = 0, \tag{11.6}$$

$$\lim_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}} \bigl[G_{\mathbf{f}}(\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}}) - G_{\mathbf{f}}(\boldsymbol{\lambda}^{\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}})\bigr] = 0. \tag{11.7}$$

The existence of those $\beta^*_{\mathbf{K}}$ follows from Theorem 6.7.

Then, combining (11.1) and (11.7) gives (6.3) as required, while (11.6) together with the property of strong fundamentality of $(\boldsymbol{\lambda}^{\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$ shows that $(\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$ is strongly fundamental as well. Hence, by Theorem 7.4 and Lemma 8.5, the vague cluster set of $(\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$ is nonempty and for every its element $\boldsymbol{\lambda}$ the following assertions both hold: $\boldsymbol{\lambda} \in \mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $\boldsymbol{\lambda}^{\beta_{\mathbf{K}}\boldsymbol{\sigma}_{\mathbf{K}}}_{\mathbf{K}} \to \boldsymbol{\lambda}$ strongly. To complete the proof, it is enough to show that $\boldsymbol{\lambda} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, but this can be done in the same way as at the end of the proof of Theorem 6.7. □

12. On the sharpness of condition (c) in Theorem 6.2

Given a closed set $F \subset \mathbb{R}^n$, for brevity let $\mathfrak{M}^1(F)$ denote the collection of all probability measures supported by F. The examples below illustrate the sharpness of condition (c) in Theorem 6.2.

12.1. Examples

Example 1. In $\mathbb{R}^n$, $n \geqslant 2$, consider the Riesz kernel $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$ of order α, $\alpha \in (0,2]$, $\alpha < n$, and a condenser $\mathbf{A} = (A_1, A_2)$, where $I^+ = \{1\}$, $I^- = \{2\}$, A_1 is compact, and $C_{\kappa_\alpha}(A_i) > 0$ for $i = 1,2$. Also consider the α-Green kernel $g^\alpha_{A_2^c}$ of $A_2^c := \mathbb{R}^n \setminus A_2$, defined by (see, e.g., [17, Chap. 4, § 5])

$$g^\alpha_{A_2^c}(x,y) := \kappa_\alpha(x,\varepsilon_y) - \kappa_\alpha(x, \beta^\alpha_{A_2}\varepsilon_y),$$

where ε_y is the (unit) Dirac measure at y and $\beta^\alpha_{A_2}$ is the operator of Riesz balayage onto A_2. Further, let $K \subset (A_1 \cup A_2)^c$ be a compact set with $C_{\kappa_\alpha}(K) > 0$, and let θ denote the (unique) minimizer in the minimal α-Green energy problem

$$\inf_{\nu \in \mathfrak{M}^1(A_1 \cup K)} g^\alpha_{A_2^c}(\nu,\nu);$$

then it holds true that

$$\|\theta\|^2_{g^\alpha_{A_2^c}} = \|\theta - \beta^\alpha_{A_2}\theta\|^2_{\kappa_\alpha} = \bigl[C_{g^\alpha_{A_2^c}}(A_1 \cup K)\bigr]^{-1}.$$

Assume, moreover, that $\mathbf{f}$ satisfies Case II with $\zeta = \theta_K$, where θ_K is the trace of θ upon K, and let $a_1 = \theta(A_1)$, $a_2 = 1$, $g_1 = g_2 = 1$. Also assume, for simplicity, A_2^c to be connected.

Proposition 12.1. *Under the above notation and requirements, the following two assertions are equivalent:*

(i) *One can choose a strictly increasing sequence of positive numbers R_k, $k \in \mathbb{N}$, and measures $\omega_k \in \mathfrak{M}^1\bigl(A_2 \cap \{R_k \leqslant |x| \leqslant R_{k+1}\}\bigr)$ so that*

$$\mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \varnothing, \tag{12.1}$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2)$ is a constraint with the components

$$\sigma^1 := \theta_{A_1} \quad \textit{and} \quad \sigma^2 := \beta^\alpha_{A_2}\theta + \bigl[1 - \beta^\alpha_{A_2}\theta(A_2)\bigr] \sum_{k \in \mathbb{N}} \omega_k. \tag{12.2}$$

(ii) $C_{\kappa_\alpha}(A_2) = \infty$*, though A_2 is α-thin at $\infty_{\mathbb{R}^n}$.*

Recall that a closed set $F \subset \mathbb{R}^n$ is *α-thin at $\infty_{\mathbb{R}^n}$* if the origin $x = 0$ is an α-irregular point for the inverse of F relative to the unit sphere (see [4]; cf. also [3, 17]). See, e.g., [17, Chap. V] for the notion of α-regularity in case $\alpha \in (0,2)$.

Example 2. With the notation and the requirements of Example 1, let $\kappa_2(x,y)$ be the Newtonian kernel $|x-y|^{-1}$ in $\mathbb{R}^3$, and let A_2 be a rotational body consisting of all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $q \leqslant x_1 < \infty$ and $0 \leqslant x_2^2 + x_3^2 \leqslant \rho(x_1)$, where $q \in \mathbb{R}$ and $\rho(x_1)$ approaches 0 as $x_1 \to \infty$. Consider the following three cases:

$$\rho(r) = r^{-s}, \quad \text{where} \quad s \in [0,\infty), \tag{12.3}$$

$$\rho(r) = \exp(-r^s), \quad \text{where} \quad s \in (0,1], \tag{12.4}$$

$$\rho(r) = \exp(-r^s), \quad \text{where} \quad s > 1. \tag{12.5}$$

As has been shown in [23], A_2 is not 2-thin at $\infty_{\mathbb{R}^3}$ in case (12.3), has finite (Newtonian) capacity in case (12.5), and it is 2-thin at $\infty_{\mathbb{R}^3}$ though has infinite (Newtonian) capacity in case (12.4). Consequently, assertion (i) from Proposition 12.1 on the unsolvability of the corresponding constrained problem holds in case (12.4), and it fails to hold in both cases (12.3) and (12.5).

12.2. Proof of Proposition 12.1

Our arguments are based on Theorem 4 from [24], which asserts that, if $F \subset \mathbb{R}^n$ is closed, $\nu \geqslant 0$ is concentrated in F^c and if, for simplicity, F^c is connected, then

$$\beta_F^\alpha \nu(\mathbb{R}^n) = \nu(\mathbb{R}^n) \iff F \text{ is not } \alpha\text{-thin at } \infty_{\mathbb{R}^n}. \tag{12.6}$$

Fix an arbitrary $\boldsymbol{\mu} = (\mu^1, \mu^2) \in \mathcal{E}^+(\mathbf{A})$. Then, by (5.10),

$$G_{\mathbf{f}}(\boldsymbol{\mu}) = \|R\boldsymbol{\mu} + \theta_K\|^2_{\kappa_\alpha} - \|\theta_K\|^2_{\kappa_\alpha} = \|\mu^1 + \theta_K - \mu^2\|^2_{\kappa_\alpha} - \|\theta_K\|^2_{\kappa_\alpha}.$$

Since, by known facts from the Riesz and α-Green potential theory [17],

$$\|\mu^1 + \theta_K - \mu^2\|^2_{\kappa_\alpha} \geqslant \|\mu^1 + \theta_K - \beta^\alpha_{A_2}(\mu^1 + \theta_K)\|^2_{\kappa_\alpha} = \|\mu^1 + \theta_K\|^2_{g^\alpha_{A_2^c}} \geqslant \|\theta\|^2_{g^\alpha_{A_2^c}},$$

we get

$$G_{\mathbf{f}}(\boldsymbol{\mu}) \geqslant \|\theta\|^2_{g^\alpha_{A_2^c}} - \|\theta_K\|^2_{\kappa_\alpha}, \tag{12.7}$$

where the inequality is actually an equality if and only if

$$\mu^1 + \theta_K = \theta \quad (\text{hence, } \mu^1 = \theta_{A_1}) \ \text{ and } \ \mu^2 = \beta^\alpha_{A_2}\theta.$$

Assume (i) to hold; then necessarily $C_{\kappa_\alpha}(A_2) = \infty$, for if not, we would arrive at a contradiction with Theorem 6.2. To establish (ii), assume, on the contrary, that A_2 is not α-thin at $\infty_{\mathbb{R}^n}$. Then, by (12.6),

$$\beta^\alpha_{A_2}\theta(A_2) = 1 \tag{12.8}$$

and consequently, by (12.2),

$$\sigma^1 = \theta_{A_1} \quad \text{and} \quad \sigma^2 = \beta^\alpha_{A_2}\theta. \tag{12.9}$$

It follows from (12.8) and (12.9) that $\boldsymbol{\sigma} \in \mathcal{E}^+_{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and inequality (12.7) for $\boldsymbol{\mu} = \boldsymbol{\sigma}$ is actually an equality. Thus, $\boldsymbol{\sigma} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, which is impossible by (12.1).

Now, assume (ii) to hold. Since A_2 is α-thin at $\infty_{\mathbb{R}^n}$, from (12.6) we get

$$c := 1 - \beta^\alpha_{A_2}\theta(A_2) > 0. \tag{12.10}$$

Choose a strictly increasing sequence $(R_k)_{k\in\mathbb{N}}$ with the property $C_{\kappa_\alpha}\bigl(A_2^{(k)}\bigr) > k$, where $A_2^{(k)} := A_2 \cap \{R_k \leqslant |x| \leqslant R_{k+1}\}$, which is possible because of the assumption $C(A_2) = \infty$, and let ω_k minimize $\kappa_\alpha(\nu, \nu)$ among all $\nu \in \mathfrak{M}^1\bigl(A_2^{(k)}\bigr)$. Then

$$\lim_{k\to\infty} \|\omega_k\|^2_{\kappa_\alpha} = 0 = \inf_{\nu \in \mathfrak{M}^1(A_2)} \kappa_\alpha(\nu, \nu),$$

which yields by standard arguments that $(\omega_k)_{k\in\mathbb{N}}$ is a strong Cauchy sequence in $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$. Since $\omega_k \to 0$ vaguely, in view of the perfectness of κ_α we thus get

$$\omega_k \to 0 \quad \text{strongly}.$$

Consider $\boldsymbol{\mu}_k = (\mu_k^1, \mu_k^2)$, $k \in \mathbb{N}$, with $\mu_k^1 = \theta_{A_1}$ and $\mu_k^2 = \beta_{A_2}^\alpha \theta + c\omega_k$, where c is given by (12.10). Then $\boldsymbol{\mu}_k \in \mathcal{E}_{\boldsymbol{\sigma}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$, where $\boldsymbol{\sigma}$ is defined by (12.2), and

$$\begin{aligned}\|\mu_k^1 + \theta_K - \mu_k^2\|_{\kappa_\alpha}^2 &= \|\theta - \beta_{A_2}^\alpha \theta - c\omega_k\|_{\kappa_\alpha}^2 \\ &= \|\theta\|_{g_{A_2^c}^\alpha}^2 + c^2\|\omega_k\|_{\kappa_\alpha}^2 - 2c\kappa_\alpha(\omega_k, \theta - \beta_{A_2}^\alpha \theta).\end{aligned}$$

Letting here $k \to \infty$, we then obtain

$$\lim_{k\to\infty} G(\boldsymbol{\mu}_k) = \|\theta\|_{g_{A_2^c}^\alpha}^2 - \|\theta_K\|_{\kappa_\alpha}^2$$

and so, by (12.7), $(\boldsymbol{\mu}_k)_{k\in\mathbb{N}} \in \mathbb{M}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Since $\boldsymbol{\mu}_k \to \boldsymbol{\gamma} := (\theta_{A_1}, \beta_{A_2}^\alpha \theta)$ strongly and vaguely, we get $\boldsymbol{\gamma} \in \mathfrak{C}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Moreover, in view of the strict positive definiteness of the Riesz kernel, $\boldsymbol{\gamma}$ is the only element of the class $\mathfrak{C}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ (see assertion (ii) of Lemma 8.3). However, because of (12.10), $\boldsymbol{\gamma} \notin \mathfrak{S}_{\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. This proves (12.1) and, hence, (i). □

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